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Lecture note on

FLUID MECHANICS

(for undergraduate students)

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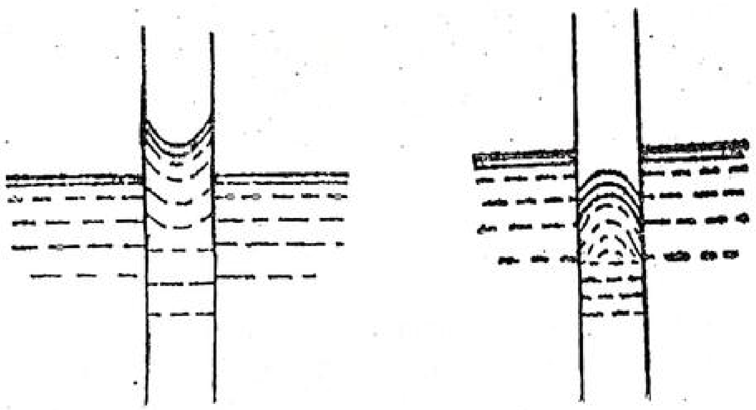

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Chapter 1

FLUID PROPERTIES

1.1 FLUID MECHANICS AND ITS BRANCHES

Fluid mechanics is that branch of engineering science which deals with the behavior of fluids at rest and in motion. The study takes into account the various properties of fluids and their effects on the resulting flow patterns, in addition to the forces acting between the fluid and its boundaries. To explain observed fluid behavior and to predict fluid behavior, the study and application of fundamental laws (conservation of mass, energy and momentum) is essential.

Fluid mechanics may be divided into three branches: (i) **fluid statics**, which deals with the cases in which the fluid is at rest and the forces on a fluid element are in equilibrium, (ii) **fluid kinematics**, which deals with the velocities, accelerations and the patterns of flow without considering forces or energy, and (iii) **fluid dynamics**, which deals with the relations between velocities and accelerations of fluid with the forces or energies causing them.

1.2 DEVELOPMENT OF FLUID MECHANICS

Until the turn of this century the study of fluids was undertaken essentially by two groups–mathematicians and engineers. The mathematicians had attempted to obtain solutions of many problems of fluid motion on purely theoretical basis, assuming an imaginary ideal (frictionless or non-viscous) fluid. The results of such studies, without consideration of all the properties of real fluids, although very useful under certain circumstances and conditions, are of limited practical use. The body of knowledge thus gained is classified under the subject of **hydrodynamics.**

On the other hand, engineers (hydraulicians) worked on empirical lines. They turned to numerous laboratory tests (experiments) and field observations on fluid flow. From these data, they developed empirical formulas in order to solve the everyday problems of fluid flow. This branch of engineering was given the name **hydraulics**.

Empirical hydraulics was confined largely to water and was limited in scope. With developments in aeronautics, chemical engineering and the petroleum industry, the need arose for a broader treatment. It was realized that the two sciences developed by two basic schools of thought working separately should come together on a common platform for a common purpose, i.e. there should be close relationship between theory and experiment. In 1904 German Professor Ludwig Prandtl introduced the concept of a boundary layer which brought together the two independent subjects. Prandtl's boundary layer theory explained the differences in the behavior of real fluids as observed by the hydraulicians and the predictions from the theory of ideal fluids by classical hydrodynamists. The combination of theoretical methods and experimental observations gave rise to rapid developments in the study of fluid flow phenomenon in many fields of engineering and this new subject came to be known as fluid mechanics.

It became clear to such eminent investigators as Reynolds, Froude, Prandtl and von Karman that the study of fluids must be a blend of theory and experimentation. Such was the beginning of the science of fluid mechanics as it is known today. Our modern research and test facilities employ mathematicians, physicists, engineers and skilled technicians, who working in teams, bring both viewpoints in varying degrees to their work.

1.3 HISTORICAL BACKGROUND

The application of fluid mechanics began in connection with the motion of stones, spears and arrows. Ships with sails were used as early as 3000 B.C. Irrigation systems have been found in prehistoric ruins in both Egypt and Mesopotamia. Aristotle (4th century B. C.)

studied the motions of bodies in thin media and in voids. Archimedes (3rd century B. C.) formulated the well-known laws of floating bodies.

The Roman aqueducts were build in the 4th century B.C. Leonardo da Vinci (1452-1519) correctly described many flow phenomena. Galileo (1564-1642) contributed much to the science of mechanics.

The Italian school of hydraulics included Gastelli (1577-1644), Torricelli (1608-1647) and Guglielmini (1655-1710), and ideas concerning the steady flow continuity equation in rivers, flow from a container, the barometer and some qualitative concepts of flow resistance in rivers came from them. In addition to his well-known laws of motion, Newton (1642-1727) proposed that fluid resistance is proportional to the velocity gradient and he also made experiments on the drag of spheres.

The mathematical science of fluid mechanics – hydrodynamics – was introduced by four 18th century mathematicians: Daniel Bernoulli and Leonhard Euler (Swiss) and Clairaut and d'Alembert (French). These were followed by Lagrange (1736-1813), Laplace (1749-1827) and an engineer, Gerstner (1756-1832) who investigated ideas on surface waves.

Experimentalists of the eighteenth century added much. These men included de Pitot, who developed a tube for measuring velocities; Chezy, who developed a resistance formula for open channels; Borda, who performed experiments related to the flow through orifices; Bossut, who built a towing tank and Venturi, who experimented with flow in changing cross-sections.

In the 19th century, the Frenchman Coulomb (1736-1806) conducted tests and drew conclusions regarding flow resistance; the German brothers Ernst (1795-1878) and Wilhelm Weber (1804-1891) conducted tests on wave motion; the French engineers Burdin (1790-1873), Fourneyman (1802-1867), Coriolis (1792-1843) and the American engineer Francis (1815-1892) contributed toward the development of hydraulic turbines; the Scotsman Russel (1808-1882) conducted tests on waves; the German Hagen (1797-1889), the Frenchman Poiseuille (1799-1869) and the Sazon Weisbach (1806-1871) did extensive work on pipe flow; the Frenchman Saint-Venant (1797-1886) contributed to open channel hydraulics; the Frenchman Dupuit (1804-1866), Bresse (1822-1883) and Basin (1829-1917) and the Irishman Manning (1816-1897) did extensive work in open channel hydraulics; the Frenchman Darcy (1803-1858) did work on pipe flow; and the Englishman William Froude (1810-1879) and his son Robert Froude (1846-1924) did extensive ship model testing. Osborne Reynolds in 1883 successfully conducted an experiment to distinguish between stream line flow and turbulent flow.

Classical and applied hydrodynamics were improved considerably during the 19th century by Navier (1785-1876), Cauchy (1789-1857), Poisson (1781-1840), Saint-Venant (1797-1886) and Boussinesq (1842-1929) in France; Stokes (1819-1903), Lord Rayleigh (1842-1919) and Lamb (1849-1934) in Great Britain; Helmholtz (1821-1894) and Kirchoff (1824-1887) in Germany.

At the end of the nineteenth century, theoretical hydrodynamics, based on Euler's equations of motion for an ideal (non-viscous) fluid, had reached a comparatively high level of development. It did not explain, however, many observed effects, such as the pressure drop in pipes and thus practicing engineers developed their own empirical science of hydraulics. These two fields - hydrodynamics and hydraulics - had much too little in common at that time. In 1904 Prandtl (1875-1953) in Germany introduced the concept of a boundary layer, a thin region adjacent to a boundary, in which the viscous effects were dominant. This proved to be the concept which unified modern fluid mechanics: aerodynamics, hydraulics, gas dynamics and convective heat transfer. It explained the differences in the behavior of real fluids as observed by hydraulicians and the predictions from the theory of non-viscous fluids by classical hydrodynamicists. **Prandtl is properly considered to be the father of modern fluid mechanics.**

Progress during the 20th century has included both analytical and experimental studies of boundary layer flow, turbulence structure, flow stability, multiphase flows and heat transfer in flowing fluids.

1.4 SCOPE OF FLUID MECHANICS

Knowledge and understanding of the basic principles and concepts of fluid mechanics are essential to analyze any system in which a fluid is the working medium. The design of virtually all means of transportation requires application of the principles of fluid mechanics. Included are aircraft for both subsonic and supersonic flight, ground effect machines, hovercraft, vertical takeoff and landing aircraft requiring minimum runway length, surface ships, submarines and automobiles. In recent years automobile manufacturers have given more consideration to aerodynamic design. This has been true for some time for the designers of both racing cars and boats. The design of propulsion systems for space flight as well as for toy rockets is based on the principles of fluid mechanics. The collapse of the Tacoma Narrows Bridge in 1940 is the evidence of the possible consequences of neglecting the basic principles of fluid mechanics. It is commonplace today to perform model studies to determine the aerodynamic forces on and flow fields around buildings and structures. These include studies of skyscrapers, baseball stadiums, smokestacks and shopping plazas.

The design of all types of fluid machinery including pumps, fans, blowers, compressors and turbines clearly requires knowledge of the basic principles of fluid mechanics. Lubrication is an application of considerable importance in fluid mechanics. Heating and ventilating systems for private homes, large office buildings and underground tunnels and the design of pipeline systems are further examples of technical problem areas requiring knowledge of fluid system. It is not surprising that the design of blood substitutes, artificial hearts, heart-lung machines, breathing aids and other such devices must rely on the basic principles of fluid mechanics.

Even some of our recreational endeavors are directly related to fluid mechanics. The slicing and hooking of golf balls can be explained by the principles of fluid mechanics.

Fluid mechanics is not a subject studied for purely academic interest; it is a subject with widespread importance both in our everyday experiences and in modern technology.

1.5 DEFINITION OF A FLUID AND ITS TYPE

A fluid is a substance which is capable of flowing and consists of liquids and gases and vapors. Liquids possesses a definite volume which only slightly changes with temperature and pressure and are practically incompressible. On the other hand, gases and vapors possess no definite volume, are easily compressed and their volumes and pressures are easily susceptible to changes in temperature.

From the viewpoint of mechanics, a fluid is a substance that deforms continuously under the application of a shear (tangential) stress no matter how small the shear stress may be and fluid motion results. Thus, **the fluids at rest cannot sustain a shear stress.** By contrast, a solid undergoes a definite displacement when subjected to a shear stress up to a certain limit (if it does not break completely) and regains its original shape and position as soon as the external shear stress is removed. Hence, a solid can withstand tensile, compressive and shearing stresses up to certain limits.

The deformations of a solid (Fig. 1.1a) and a fluid (Fig. 1.1b) under the action of a constant shear force F are considered. In Fig. 1.1a, a the shear force is applied to the solid through the upper of two plates to which the solid has been bonded. When the shear force F is applied to the plate, the block is deformed as shown. From our previous work in mechanics, we know that, provided the elastic limit of the solid material is not exceeded, the deformation is proportional to the applied shear stress, $\tau = F/A$, where A is the area of the surface in contact with the plate.

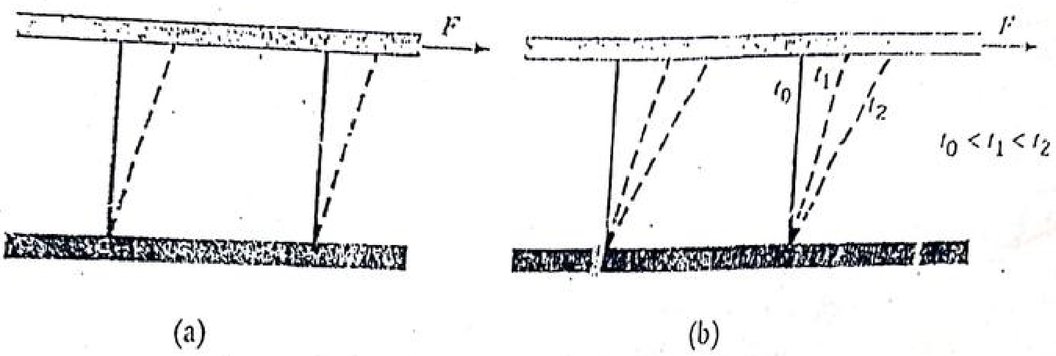


Fig. 1.1 Behavior of a solid and a fluid under the action of a constant shear force

To repeat the experiment with a fluid between the plates, use a dye marker to outline a fluid element as shown by the solid lines (Fig. 1.1b). When the shear force F is applied to the upper plate, the deformation of the fluid element continues to increase as long as the force is applied. The fluid in direct contact with the solid boundary has the same velocity as the boundary itself; there is no slip at the boundary (no-slip condition). The shape of the fluid element at successive instants of times $t_0 < t_1 < t_2$, is shown (Fig.1.1b) by the dashed lines, which represent the positions of the dye markers at successive times.

A fluid may be ideal or real. An ideal fluid is a fluid in which there is no friction, i.e., its viscosity is zero. No resistance is offered when such a fluid flows. No such fluid exists in nature. Real fluids are those which have viscosity, surface tension and compressibility in addition to its density. In a real fluid some shear stress will exist and some frictional work will be done. The assumption of ideal fluid helps in simplifying the mathematical analysis.

1.6 FLUID AS A CONTINUUM

A fluid is composed of a very large number of molecules continuously agitating and colliding and the fluid properties like density, pressure, temperature, etc. are the outcome of the interactions of large number of molecules. In dealing with fluid-flow relations on a mathematical or analytical basis, it is necessary to consider that the actual molecular structure is replaced by a hypothetical continuous medium, called the continuum. In most problems of fluid mechanics, the size of the body through which flow takes place is quite vast compared to the size of molecules and the flow contains a great many molecules per unit volume. So at ordinary conditions the fluids may be considered as continuous in which all properties like density, viscosity, specific volume, velocity, acceleration, pressure, temperature, etc. vary continuously through a fluid with respect to time and space.

1.7 DENSITY AND SPECIFIC WEIGHT

The density or mass density or specific mass of a fluid is its mass per unit volume. It is given by

$$\rho = \frac{M}{V} \tag{1.1}$$

where ρ is the density (kg/m^3), M is the mass (kg) and V is the volume (m^3). The density is a fluid property, since it depends on mass which is independent of location. The density varies with change of temperature and pressure. The density of water at 4^{0}C and atmospheric pressure is 1000 kg/m^3. The density of sea water having a salinity of 3.3% is 1026 kg/m^3.

The specific weight or weight density of a fluid is the weight per unit volume. It is given by

$$\gamma = \frac{W}{V} = \frac{Mg}{V} = \rho g \tag{1.2}$$

where γ is the specific weight (N/m^3), W is the weight (N) and g is the acceleration due to gravity (= 9.81 m/s^2). Specific weight of fresh water at 4^0C and at mean sea level is 9810 N/m^3.

The specific weight is not a fluid property since it depends on the acceleration due to gravity which change from place to place on the earth and on the other planets. For liquids, the effect of small pressure change on the specific weight is practically negligible, but it changes considerably with variation of temperature. For gases, the temperature as well as the pressure affect the values of specific weight considerably. Therefore, the specific weight of gases is always given with respect to some specified temperature and pressure.

Example 1.1

A certain liquid weighs 29.42 kN and occupies 4 m^3. Determine its specific weight and mass density.

Solution Specific weight, $\gamma = \frac{W}{V} = \frac{29.42 \times 1000}{4} = 7355\ N/m^3$

Mass density, $\rho = \frac{\gamma}{g} = \frac{7355}{9.81} = 749.75\ kg/m^3$

1.8 VISCOSITY

Of all the fluid properties, viscosity requires the greatest consideration in the study of fluid flow. Viscosity is the property of a fluid by virtue of which it offers resistance to shear or angular deformation. So, the viscosity of a fluid is a measure of its resistance to flow. It is the property of real fluids which distinguishes them from ideal or non-viscous fluids.

For a highly viscous fluid at low velocities, the fluid flows in parallel layers and for this type of flow, Newton postulated that the shear stress on an interface tangent to the direction of flow is proportional to the spatial rate of change of velocity (or velocity gradient) normal to the flow (Fig.1.2). Mathematically

$$\tau \propto \frac{du}{dy} \quad \text{or} \quad \tau = \mu \frac{du}{dy} \qquad (1.3)$$

where the constant of proportionality μ is known as the coefficient of viscosity or dynamic viscosity or absolute viscosity or simply viscosity. Equation (1.3) is the Newton's law of viscosity. The SI units of viscosity are Pa.s or $N.s/m^2$ or kg/m.s. In the CGS system of units, the unit of viscosity is poise and 1 poise = 0.1 $N.s/m^2$. The viscosity of water at 20^0C is 1 centipoise or 0.001 $N.s/m^2$.

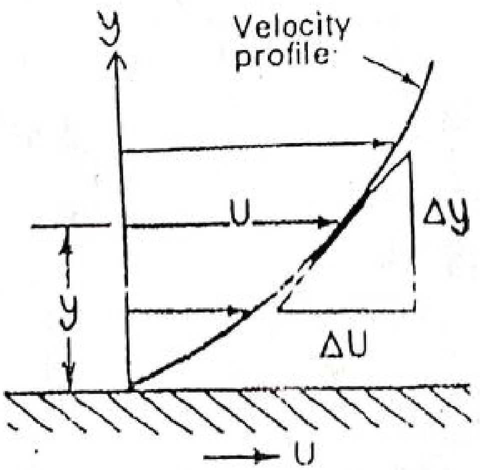


Fig.1.2 Velocity profile and velocity gradient

The kinematic viscosity ν is defined as the ratio of dynamic viscosity to density, i.e.

$$\nu = \frac{\mu}{\rho} \tag{1.4}$$

and its SI units are m^2/s. In the CGS system of units, the unit of viscosity is stoke and 1 stoke $= 1\ cm^2/s$ or $10^{-4}\ m^2/s$.

Derivation of the Newton's law of viscosity: In Fig. 1.3 a fluid is placed between two closely spaced parallel plates. The lower plate is fixed and a shear force F is applied at the upper plate, which exerts a shear stress $\tau = F/A$ on the fluid between the plates, where A is the area of the upper plate. The force F causes the upper plate to move with a steady non-zero velocity U. The fluid in the area *abcd* flows to the new position *ab'c'd*, each fluid particle moving parallel to the plate and the velocity u varying uniformly from zero at the stationary plate to U at the upper plate. Experiments show that, other quantities being held constant, F is directly proportional to A and to U and inversely proportional to thickness t. In equation form

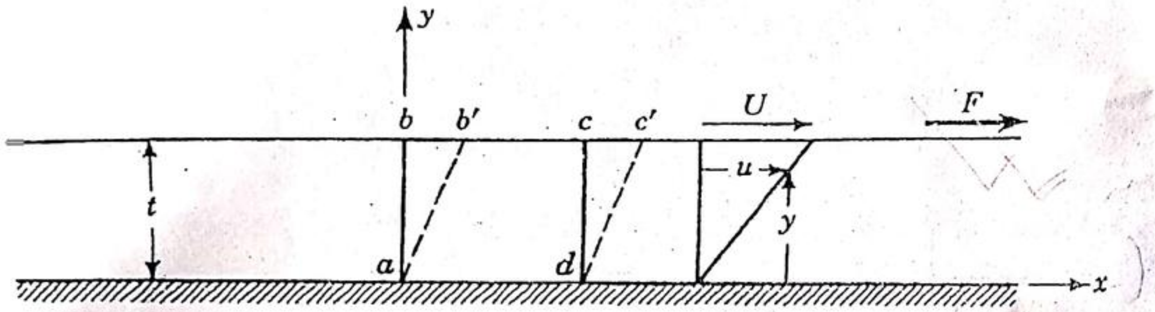


Fig. 1.3 Deformation resulting from application of constant shear force

$$F \propto \frac{AU}{t} \quad \text{or} \quad F = \mu\frac{AU}{t} \tag{1.5}$$

in which μ is the proportionality factor and includes the effect of the particular fluid. Writing $\tau = F/A$ for the shear stress,

$$\tau = \mu\frac{U}{t}$$

The ratio U/t is the angular velocity of the line ab, as it is the rate of angular deformation of the fluid, i.e. the rate of decrease of angle *bad*. The angular velocity may also be written as du/dy, as both U/t and du/dy express the velocity change divided by the distance over which the change occurs. However, du/dy is more general, as it holds for the situations in which the angular velocity and shear stress change with y. The velocity gradient du/dy may also be visualized as the rate at which one layer moves relative to the adjacent layer. In differential form

$$\tau = \mu\frac{du}{dy}$$

is the relation between the shear stress and the velocity gradient and known as the Newton's law of viscosity.

Newtonian and non-Newtonian Fluids: Fluids may be classified as Newtonian and non-Newtonian. The fluids which follow the Newton's law of viscosity, given by Eq.(1.3), are known as the Newtonian fluids. For these fluids there is a linear relationship between the magnitude of applied shear stress (Fig.1.4) and the resulting rate of deformation and μ is constant. Gases and thin liquids like water, kerosene, air, etc. tend to be Newtonian. The fluids which do not follow the Newton's law of viscosity, given by Eq.(1.3), are known as the non-Newtonian fluids. Thick, long-chained hydrocarbons like suspensions, solution of graphites, solution of polymers, blood, etc. may be non-Newtonian.

Some substances which flow only after the application of some finite stress are known as plastics. An ideal plastic has a definite yield stress and a constant linear relation of τ to du/dz (μ constant)(Fig. 1.4). A thixotropic substance, such as printer's ink, has a viscosity that is dependent upon the immediately prior angular deformation of the substance (Fig. 1.4).

An ideal (non-viscous) fluid has no viscosity. With zero viscosity the shear stress is always zero, regardless of the motion of the fluid. An ideal fluid plots as the ordinate in Fig. 1.4

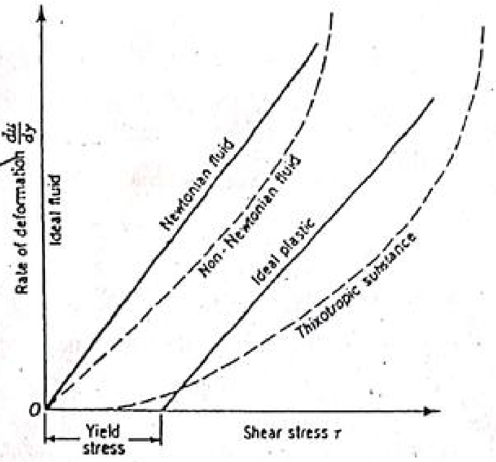


Fig. 1.4 Viscous behavior of fluids

Effect of temperature on viscosity: The viscosity of liquids decreases but that of gases increases with the increase in temperature. The viscosity of liquids is mainly due to intermolecular cohesion which decreases with the increase in temperature. In gases the molecules are more free to move as the intermolecular forces between adjacent layers are nearly negligible. Thus, there is a transfer of momentum between different layers of gases. That is why, the viscosity of gases is mainly due to transfer of momentum of molecules. When temperature is raised, the molecular activity increases. Consequently, more transfer of momentum takes place, as a result the viscosity of gas is increased with increase in temperature.

Example 1.2

The specific gravity of water at 20^0C is 0.998 and its viscosity is 0.001 N.s/m^2. Determine its kinematic viscosity.

Solution For water at 20^0C, $\mu = 0.001$ N.s/m^2 and $\rho = 0.998 \times 1000 = 998$ kg/m^3

$$\therefore \ \nu = \frac{\mu}{\rho} = \frac{0.001}{998} = 1.002 \times 10^{-6} \ m^2/s$$

1.9 COMPRESSIBILITY

Fluids may be compressed by an external pressure applied to a volume of fluid. Gases, however, are compressed by an external pressure to a much greater degree than liquids.

The compressibility of a fluid is inversely proportional to its bulk modulus of elasticity. Consider that a force is applied to any fluid so that the volume of the fluid V decreases by an amount ΔV. The decrease per unit volume is proportional to the force per unit area (i.e. pressure) of the fluid. So

$$p \propto \frac{-dV}{V}$$

or

$$p = -K\frac{dV}{V}$$

or

$$K = -\frac{p}{dV/V} \tag{1.6}$$

where K is the volume or bulk modulus of elasticity which has the dimensions of pressure, i.e. force per unit area (N/m^2). The value of K for water at any temperature increases with pressure. At 20^0C, it is about 2.19×10^9 N/m^2 at atmospheric pressure and it increases linearly to 2.86×10^9 N/m^2 at a pressure of 1000 atmospheres. In this range of pressure at 20^0C

$$K = (2.19\times 10^9 + 6.7p)\ N/m^2 \tag{1.7}$$

where p is the gage pressure in N/m^2. Note that 1 atmosphere = 10^5 N/m^2 =10^5 Pa.

Water is about 80 times as compressible as steel. It has a minimum compressibility at about 50^0C.

For most practical problems, water can be considered as incompressible. But in case of water flowing through pipes, when the sudden or larger changes in pressure occurs as in water hammer, the compressibility cannot be neglected.

In fluid mechanics, compressibility is considered mainly when the velocity of flow is high enough reaching 0.2 of the speed of sound in the medium.

Example 1.3

The decrease in volume of a certain mass of liquid is observed to be 1/500 of its original volume when a pressure of 5000 kN/m^2 is applied on it. Find the bulk modulus of elasticity of the liquid.

Solution Here, $\Delta V/V = -1/500$, p = 5000 kN/m^2

$$\therefore K = -\frac{p}{\Delta V/V} = -\frac{5000\times 1000}{-1/500} = 2.5\times 10^9\ N/m^2$$

1.10 COHESION AND ADHESION

Cohesion and adhesion are properties of fluids which are the two forms of molecular attraction. Cohesion is the property of a fluid by which molecules of the same fluid are attracted. This property enables the liquids to resist tensile stress.

Adhesion is a property of a fluid by which molecules of different liquids are attracted to each other or molecules of a liquid are attracted to another body. This enables the two different liquids to adhere to each other or to a liquid to adhere to another body.

1.11 SURFACE TENSION

The surface tension is the property of a liquid which enables it to resist tensile stress. At the interface between a liquid and a gas or between two immiscible liquids, a film or special layer seems to form on the liquid which is capable of resisting a small tensile stress. This is due primarily to molecular attraction between like molecules (cohesion) and molecular attraction between unlike molecules (adhesion). In the interior of a liquid the cohesive forces cancel, but at the free surface the liquid cohesive forces from below exceed the adhesive forces from the gas. So there is a tendency for the liquid molecules to be drawn into the interior and for the surface to contract to a minimum area giving rise to the phenomenon known as surface tension. It is designated by σ and has the dimensions of force per unit length (N/m). This force may be considered to be in a direction normal to any line drawn on the interface and in the plane of the interface (Fig.1.5).

Surface tension shows a variety of phenomena. It enables water to support small objects like dust particles, needle, etc. Small insects can alight on and able to remain atop the water surface. The falling drops of water (e.g. rain water) and the soap bubbles become spherical. Surface tension leads to the phenomenon of capillary rise or fall in a narrow tube.

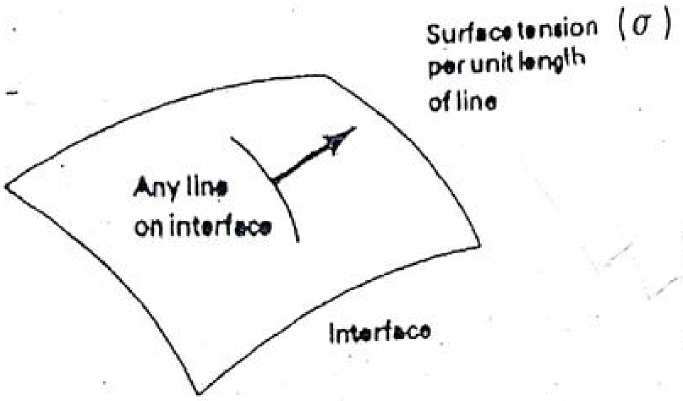
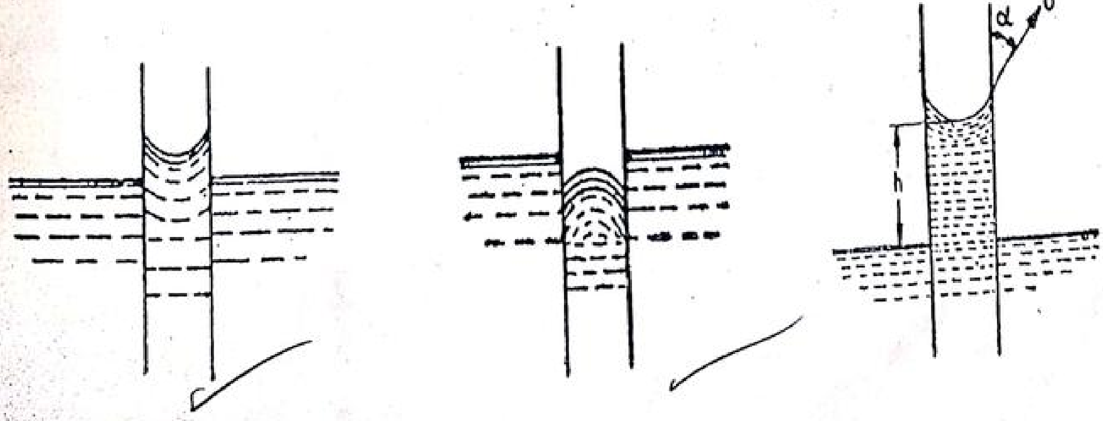


Fig. 1.5 Surface tension force on an interface

The surface tension of water in air at atmospheric pressure is about 0.073 N/m at 20^0C and decreases slightly with an increase in air temperature. Surface tension effects are generally negligible in most engineering situations. However, they may be important in problems involving capillary rise, the formation of drops and bubbles, the breakup of liquid jets and in hydraulic model studies where the model is small.

Capillarity: When a tube of small diameter is dipped in water, the water wets the tube and rises up in the tube with an upward concave surface (Fig. 1.6a). This is due to the reason that the adhesion between the tube and the water molecules is more than the cohesion between the water molecules. But when the same tube is dipped in mercury, the mercury does not wet the tube and depresses down in the tube with an upward convex surface (Fig. 1.6b). This is due to the reason that the adhesion between the tube and the mercury molecules is less than the cohesive forces between the mercury molecules. These properties of adhesion and cohesion in addition to surface tension result in the phenomenon of capillarity by which a liquid will rise or fall in a narrow tube when the tube is dipped in the liquid.



(a) In water (b) In mercury (c) Contact angle

Fig. 1.6 Capillary rise and fall

Contact angle α between a liquid and a solid is defined in Fig. 1.6(c). It depends on the cleanliness of the surface and the property of the liquid. When the contact angle is less than 90^0, the liquid tends to wet the solid surface and capillary rise results (Fig. 1.6a). When the contact angle is more than 90^0, the liquid does not wet the solid and capillary depression occurs (Fig. 1.6b). The capillary depression may affect the reading of a barometer or manometer using mercury as the gage liquid.

Capillary rise for water: Refer to Fig. 1.6(c). Let h is the height of the capillary rise, d is the diameter of the capillary tube, α is the contact angle, σ is the force of surface tension and γ is the specific weight of water. Then, the weight of the water column h which acts downward is balanced by the force exerted by the vertical component of the force of surface tension, i.e.

$$\frac{\pi}{4} d^2 h \times \gamma = \sigma \times \pi \times d \cos\alpha$$

or

$$h = \frac{4\sigma\cos\alpha}{\gamma d} = \frac{4\sigma\cos\alpha}{\rho g d} \qquad (1.8)$$

If the tube is clean, $\alpha = 0^0$ for water and about 140^0 for mercury. For water, taking $\alpha = 0^0$, σ = 0.074 N/m, ρ = 1000 kg/m^3 and g = 9.81 m/s^2, we have

$$h = \frac{4 \times 0.074 \times 1}{1000 \times 9.81 \times d}\, m = \frac{3.02 \times 10^{-5}}{d}\, m \qquad (1.9)$$

When d = 100 mm, h = 0.302 mm, when d = 10 mm, h = 3.02 mm and when d = 1 mm, h = 30.2 mm.

When the diameter of a tube or the head or the depth of flow is less than 6 mm, the effect of the surface tension force is almost negligible. As we do not encounter such small dimensions in fluid mechanics, surface tension forces are not considered.

Relationship between the pressure intensity inside a droplet and the surface tension: Consider one-half of a spherical droplet of radius r (Fig. 1.7). Let p denotes the intensity of pressure inside the droplet and σ denotes the surface tension force. Then, the force of surface tension acting outward on the circumference = $\sigma \times 2\pi r$, and the pressure force acting inward = $p \times \pi r^2$. For equilibrium, the two forces must be equal and opposite. Therefore,

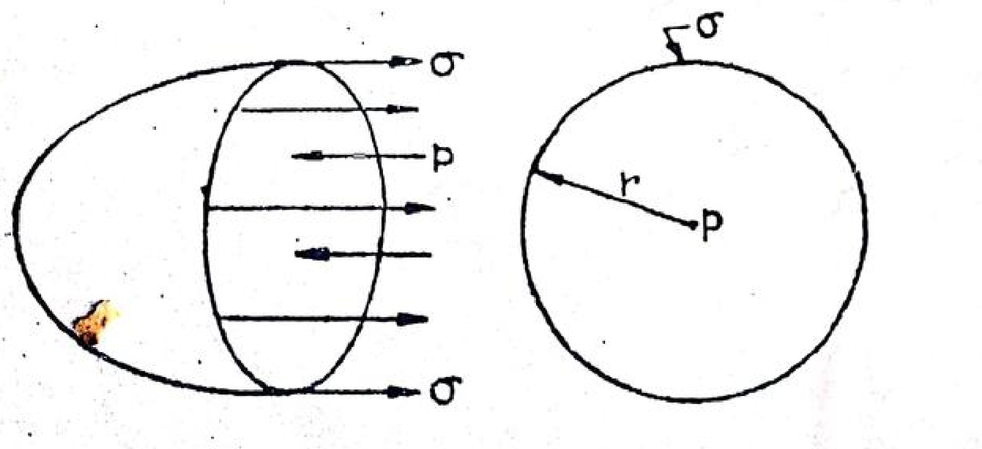


Fig. 1.7 Pressure intensity inside a droplet

$$p \times \pi r^2 = \sigma \times 2\pi r$$

$$\therefore p = \frac{2\sigma}{r} \qquad (1.10)$$

So, the pressure intensity inside a droplet increases with the decrease of the size of the droplet.

Example 1.4

A capillary tube having an inside diameter of 6 mm is dipped into water at 20^{0}C. Determine the height of capillary rise. Take $\alpha = 30^0$ and $g = 9.81$ m/s^2.

Solution From Table 1.1, we have for water at 20^{0}C, $\sigma = 0.0728$ N/m and $\rho = 998$ kg/m^3. Therefore, we obtain

$$h = \frac{4\sigma\cos\alpha}{\rho g d} = \frac{4 \times 0.0728 \times \cos 30^0}{998 \times 9.81 \times 6 \times 10^{-3}} = 4.29 \times 10^{-3}\ m = 4.29\ mm$$

PROBLEMS AND EXERCISES

1.1 Define Fluid Mechanics. What are the three branches of Fluid Mechanics?

1.2 Define a fluid from the viewpoint of mechanics.

1.3 Write short notes on (i) mass density, (ii) specific weight, (iii) absolute viscosity, (iv) dynamic viscosity, (v) Newtonian fluid, (vi) non-Newtonian fluid, (vii) compressibility/ bulk modulus of elasticity, (viii) cohesion and adhesion, (ix) surface tension, and (x) capillarity.

1.4 State Newton's law of viscosity.

1.5 Explain why the viscosity of a liquid decreases and that of a gas increases with increase in temperature.

1.6 Explain why oil, when poured on water, spreads into a very thin film on the water surface.

1.7 Write the values of (i) mass density, (ii) specific weight, (iii) absolute viscosity, (iv) dynamic viscosity, (v) bulk modulus of elasticity, and (vi) surface tension for water at 20^{0}C.

1.8 A liquid has a viscosity of 0.005 Pa.s and a density of 850 kg/m^3. Calculate its kinematic viscosity.

1.9 A liquid compressed in a cylinder has a volume of 1000 cm^3 at 1 MN/m^2 and a volume of 995 cm^3 at 2 MN/m^2. What is its bulk modulus of elasticity?

1.10 Calculate the capillary rise/drop in mm in a glass tube of 4 mm diameter when immersed in mercury at 20^{0}C. The surface tension of mercury at 20^{0}C in contact with air is 0.52 N/m, the contact angle for mercury is 130^0 and the density of mercury at 20^{0}C is 13550 kg/m^3.

1.11 An infinite plate is moved over a second plate on a layer of liquid as shown (Fig. 1.8). For small gap width d, we assume a linear velocity distribution in the liquid. The liquid viscosity is 0.65 centipoise and its specific gravity is 0.88. Determine (a) the kinematic viscosity of the liquid, (b) the shear stress on the upper plate, (c) the shear stress on the lower plate, and (d) the direction of each shear stress calculated in (b) and (c).

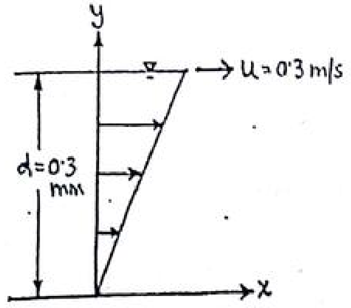


Fig. 1.8 (Prob. 1.11)

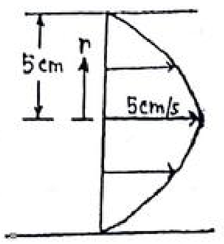


Fig. 1.9 (Prob. 1.12)

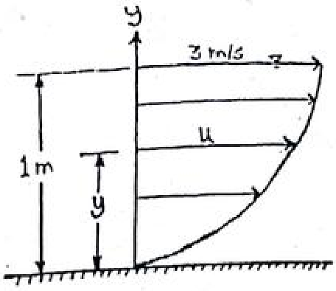


Fig. 1.10 (Prob. 1.13)

1.12 The velocity distribution in a 5 cm radius pipe is given by (Fig. 1.9)

$$u = 5\left(1 - \frac{r^2}{25}\right) \quad \text{cm/s}$$

where r is in cm. Find the shear stress at the pipe wall if the fluid has a viscosity of 2 centipoise. What is the resistance force per km length of pipe due to flow?

1.13 Assuming a velocity distribution as shown in Fig. 1.10, which is a parabola having its vertex 1 m from the boundary, calculate the velocity gradient for y = 0, 0.25, 0.50, 0.75 and 1 m. Also, calculate the shear stresses at these points if the fluid is water at 20°C.

****

Table 1.1 Properties of water

Temp. T (°C)	Density ρ (kg/m³)	Kinematic viscosity ν (m²/s)·10^{6}	Surface tension σ (N/m)	Bulk modulus of elasticity K(N/m²)·10^{9}
0	999.87	1.787	0.0757	1.98
1	999.93	1.728	0.0755	
2	999.97	1.671	0.0753	
3	999.99	1.618	0.0751	
4	1000.00	1.567	0.0749	
5	999.99	1.519	0.0748	2.03
6	999.97	1.472	0.0747	
8	999.88	1.386	0.0745	
10	999.73	1.307	0.0742	2.09
12	999.52	1.235	0.0740	
14	999.27	1.169	0.0737	2.14
16	998.97	1.109	0.0734	
18	998.62	1.053	0.0731	
20	998.23	1.002	0.0728	2.19
25	997.08	0.890	0.0720	
30	995.68	0.798	0.0712	2.25
35	994.06	0.719	0.0704	
40	992.25	0.653	0.0696	2.26
45	990.25	0.596	0.0689	
50	988.07	0.547	0.0680	2.26
60	983.24	0.467	0.0661	2.25
70	977.81	0.404	0.0643	2.22
80	971.83	0.355	0.0626	2.17
90	965.34	0.315	0.0607	
100	958.38	0.282	0.0589	

Source: Van Rijn(1990)

Note: Values of surface tension refer to the surface of water in air at atmospheric pressure

Chapter 2

FLUID STATICS

2.1 INTRODUCTION

Fluid statics considers the situations when the fluid is at rest. Hydrostatics deals with pressure of a liquid at rest. In a fluid at rest there are no shear stresses and, according to Pascal's law, the pressure p at a point within the fluid is isotropic, i.e. equal in all directions. The direction of pressure is always at right angles to the surface on which it acts.

The pressure is calculated by dividing the normal force by the area. If F represents the total force on some finite area A, while dF represents the force on an infinitesimal area dA, the pressure or the intensity of pressure or the average pressure is

$$p = \frac{dF}{dA} \qquad (2.1)$$

If the pressure is uniform over the total area, then

$$p = \frac{F}{A} \qquad (2.2)$$

In SI units, pressure is expressed in N/m^2 (Pascal) or kN/m^2.

2.2 PASCAL'S LAW

The Pascal's law states that the pressure at a point in a fluid at rest has the same magnitude in all directions. This law is applied in the construction of machines used for large forces, e.g. hydraulic press, hydraulic jack, hydraulic crane, hydraulic riveter, etc.

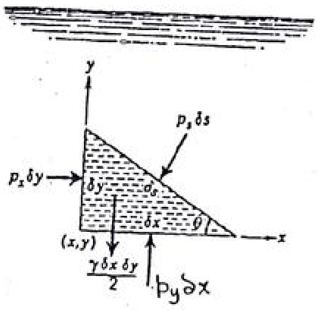


Fig. 2.1 Free-body diagram of wedge-shaped particle

To prove the Pascal's law, a small wedge-shaped element of unit width is taken at the point (x, y) in a fluid at rest (Fig.2.1). Since there can be no shear forces, the only forces are the pressure forces acting normal to the three faces and ~~the gravity force or~~ the weight of the element acting vertically downward. For the equilibrium of the element, we have in the x and y directions, the force equations

$$\sum F_x = p_x \delta y - p_s \delta s \sin\theta = 0$$

$$\sum F_y = p_y \delta x - p_s \delta s \cos\theta - \gamma \cdot \frac{\delta x \delta y}{2} = 0$$

in which p_x, p_y and p_s are the average pressures on the three faces, γ is the specific weight and ρ is its density of the fluid. Using the geometric relations

$$\delta s \sin\theta = \delta y \qquad \delta s \cos\theta = \delta x$$

the above equations simplify to

$$p_x \delta y - p_s \delta y = 0 \qquad\qquad p_y \delta x - p_s \delta x - \gamma \frac{\delta x \delta y}{2} = 0$$

The last term of the second equation is of higher order than the other two terms and may be neglected. When divided by δy and δx, respectively, the equations give

$$p_x = p_y = p_s \qquad (2.3)$$

Since θ is any arbitrary angle and the results are independent of θ, Eq. (2.3) proves that the pressure is the same in all directions at a point in a static fluid. Although the proof was carried out for a two-dimensional case, it may demonstrated for the three-dimensional case with the equilibrium equations for a small tetrahedron of fluid with three faces in the coordinate planes and the fourth face inclined arbitrarily.

2.3 PRESSURE VARIATION IN A STATIC FLUID

Consider the forces acting on an element of fluid at rest (Fig. 2.2). The element has sides δx, δy and δz in form of a parallelepiped with pressure p at its center (x, y, z). Let the pressure increases @ ∂p/∂x, ∂p/∂y and ∂p/∂z along the positive x, y and z directions, respectively. The forces acting on the element consist of surface forces and body forces. The forces which are distributed over the entire mass of the fluid are called body forces. For the element considered above, the body force is the gravity force or weight of the fluid element γδxδyδz acting vertically downward from the center of the element. Surface forces are those which act normal or tangential to a surface. In a static fluid, there is no stress (e.g. shear stress) tangential to a surface and the only force is the pressure force normal to a surface.

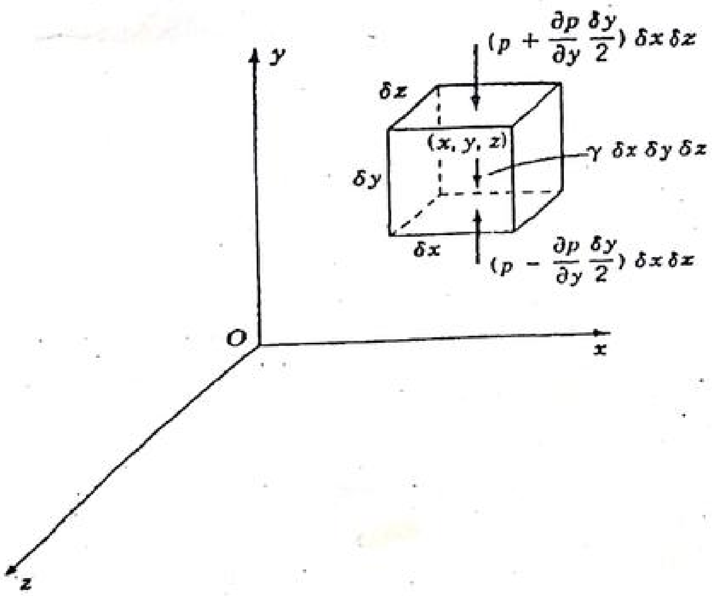


Fig. 2.2 Rectangular parallelepiped element of fluid at rest

Taking the y axis vertically upward, the force exerted on the bottom surface normal to the y axis is

$$(p - \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z$$

and the force exerted on the opposite face is

$$(p + \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z$$

where δy/2 is the distance from the center to a face normal to y. Summing the forces acting on the element in the y direction gives the resulting force in the y direction as

$$F_y = (p - \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x\delta z - (p + \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x\delta z - \gamma\delta x\delta y\delta z$$

$$= -\frac{\partial p}{\partial y}\delta x\delta y\delta z - \gamma\delta x\delta y\delta z$$

For the x and z directions, since no body forces act, we will have

$$F_x = -\frac{\partial p}{\partial x}\delta x\delta y\delta z \qquad F_z = -\frac{\partial p}{\partial z}\delta x\delta y\delta z$$

Since the fluid element is in static equilibrium, the resultant force in any direction must be zero. So, setting $F_x = F_y = F_z = 0$ since $\delta x\delta y\delta z$, which is the volume of the parallelepiped is not zero, we obtain

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = -\gamma \qquad \frac{\partial p}{\partial z} = 0 \qquad (2.4)$$

Equation (2.4) indicates that the pressure is independent of x and z and it depends on y alone and we can replace $\partial p/\partial y$ by dp/dy and so

$$\frac{dp}{dy} = -\gamma \qquad \text{or,} \quad dp = -\gamma dy \qquad (2.5)$$

Equation (2.5) is known as the fundamental equation of fluid statics. As the orientation of the element is chosen arbitrarily, it may be concluded that

i) pressure is same at every point in a horizontal plane (Pascal's law), and

ii) pressure varies in the y or vertical direction only and it decreases as y increases in the positive y direction.

For fluids which may be considered homogeneous and incompressible, γ is constant and Eq. (2.5) when integrated gives

$$p - p_1 = -\gamma(y - y_1) \qquad (2.6)$$

where p is the pressure at an elevation y. For the case of a liquid at rest, it is convenient to measure distances vertically downward from the free liquid surface. It h is the distance or depth below the free liquid surface and if the pressure of air and vapor on the surface is arbitrarily taken to be zero, Eq. (2.6) can be written as

$$p = \gamma h \qquad (2.7)$$

or

$$\frac{p}{\gamma} = h \qquad (2.8)$$

which is the hydrostatic law of variation of pressure, p is the increase in pressure from that at the free surface and h is the hydrostatic pressure head measured vertically downward from the free liquid surface. Equation (2.7) or (2.8) demonstrates that the hydrostatic pressure p varies with h.

It is obvious that the pressure can be expressed either as a force per unit area p, given by Eq.(2.7), or as a height of the equivalent liquid column h, given by Eq. (2.8).

As there must always be some pressure on the surface of any liquid, the total pressure at any depth h is given by Eq. (2.7) plus the pressure on the surface. In many situations this surface pressure may be disregarded, as pointed out in the following section.

From Eq. (2.7) it may be seen that all points in a connected body of constant density fluid at rest are under the same pressure if they are at the same depth below the free liquid surface (Pascal's law). This indicates that a surface of equal pressure for a liquid at rest is a horizontal plane. Strictly speaking, it is a surface everywhere normal to the direction of gravity and is approximately a spherical surface concentric with the earth. For practical purposes, a limited portion of this surface may be considered a plane area.

Example 2.1

Determine the intensity of pressure of oil of specific gravity 0.75 at a depth of 4 m below the free surface.

Solution We have, specific weight of oil, $\gamma = 0.75 \times 9.81 = 7.3575\ kN/m^3$, $h = 4\ m$

$\therefore$ Intensity of pressure, $p = \gamma h = 7.3575 \times 4 = 29.43\ kN/m^2$

Example 2.2

Determine the height of a water column equivalent to a pressure of 4 kN/m^2.

Solution We have, $p = 4\ kN/m^2$, $\gamma = 9.81\ kN/m^3$

$$\therefore h = \frac{p}{\gamma} = \frac{4}{9.81} = 0.41\ m$$

2.4 ABSOLUTE AND GAGE PRESSURES

The pressure measured relative to the local atmospheric or barometric pressure is known as the gage pressure. This is because practically all the pressure gages (instruments for pressure measurement) register zero when open to the atmosphere and measures the difference between the pressure of the fluid to which they are connected and the surrounding air.

When the pressure is sub-atmospheric or less than atmospheric, the gage reads less than zero. Therefore, sub-atmospheric pressure is referred to as negative gage pressure or vacuum or suction pressure. A perfect vacuum, where there are no fluid molecules at all, would correspond to absolute zero pressure.

A pressure measured with the absolute zero as a datum is called the absolute pressure. Absolute pressure has always positive value. On gage pressure scale, the local atmospheric pressure is zero. The two scales of pressure measurement, the gage scale and the absolute scale, are related by

$$p_{absolute} = p_{atmospheric} + p_{gage} \quad (2.9)$$

where p_{gage} may be positive or negative. Figure 2.3 shows the units and scales for pressure measurement.

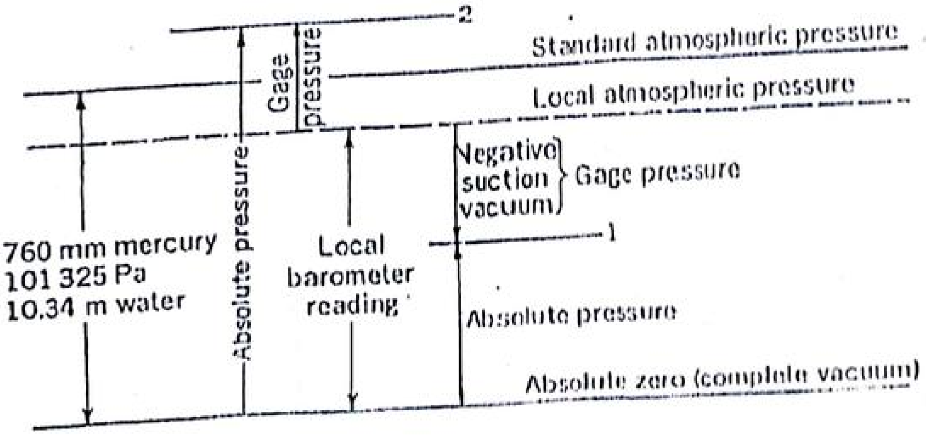


Fig. 2.3 Units and scales for pressure measurement

The standard atmospheric pressure is the mean pressure at sea level and amounts to 760 mm Hg or 101.325 kN/m^2 or 10.34 m water. The atmospheric pressure varies with altitude and, at a given place, varies from time to time because of changes in meteorological conditions.

The gage pressure is measured by a Bourdon gage. The absolute pressure of the atmosphere is measured by a mercury barometer or an Aneroid barometer.

2.5 MANOMETERS

Manometers are devices that employ liquid columns to determine pressure or difference in pressure. Manometers are of two types:

1. A simple manometer, usually called a piezometer, measures the pressure in a pipe or vessel full of liquid.
2. Differential manometer measures the difference of pressure between any two points on a pipeline or on two different pipes full of liquid.

Simple Manometer or Piezometer: The most elementary manometer, called a piezometer, is a vertical glass tube one end of which is open to the atmosphere and the other end is connected to the side of the container containing the fluid whose pressure is to determined (Fig. 2.4a). Liquid rises in the tube until equilibrium is reached. The pressure is then given by the vertical distance h from the meniscus (open liquid surface) inside the tube to the point where the pressure is to be measured, expressed in units of length of the liquid in the container. The diameter of the tube should not be less than 10 mm to avoid error due to capillary rise. If h is the height of liquid column in piezometer tube in m of liquid, the intensity of pressure at A is given by

$$p_A = \gamma h \qquad (2.10)$$

in N/m^2, where γ is the specific weight (N/m^3) of the liquid.

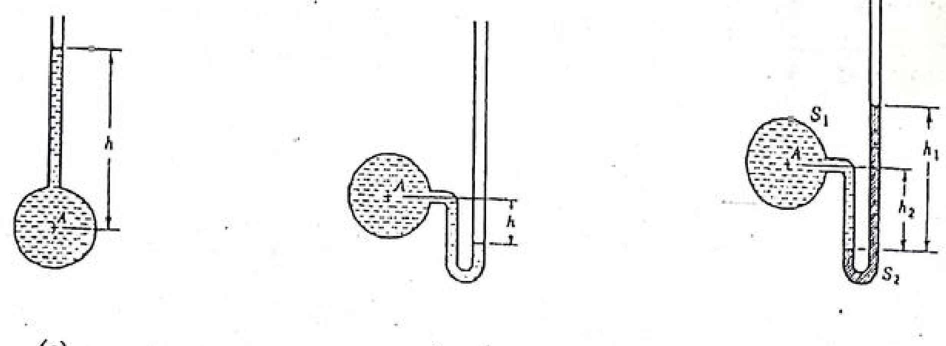


Fig. 2.4 Simple manometers: (a) Piezometer, (b) U-tube manometer, and (c) U-tube manometer with a second liquid

Double Column or U-tube Manometer: It is obvious that the piezometer would not work for negative gage pressure, because air would flow into the container through the tube. It is also impractical for measuring large pressures at A, since the vertical tube would need to be very large. If the specific gravity of the liquid is S, the pressure at A is hS units length of water.

For measurement of small negative or positive gage pressures in a liquid, a double column or U-tube manometer (Fig. 2.4b) may be conveniently used. When the gage pressure is negative, the meniscus comes to rest below A as shown. Since the pressure at the meniscus is zero gage and since pressure decreases with elevation,

$$h_A = -hS \quad \text{units of length of liquid} \tag{2.11}$$

and

$$p_A = -\gamma h_A \quad \text{N/m}^2 \tag{2.12}$$

where S is the specific gravity and γ is the specific weight of the liquid whose pressure is to be measured.

For greater negative or positive gage pressures, a second liquid of greater specific gravity is employed (Fig. 2.4c). It must be immiscible in the first liquid, which may now be a gas. If the specific gravity of the fluid A is S_1 and the specific gravity of the manometer liquid is S_2, the equation for pressure at A may be written thus, starting at either A or the upper meniscus and proceeding through the manometer,

$$h_A + h_2 S_1 - h_1 S_2 = 0 \quad \text{or} \quad h_A = h_1 S_2 - h_2 S_1 \tag{2.13}$$

and

$$p_A = \gamma h_A \tag{2.14}$$

in which h_A is the unknown pressure expressed in length units of water and h_1 and h_2 are in length units. If A contains a gas, S_1 is usually so small that $h_2 S_1$ may be neglected.

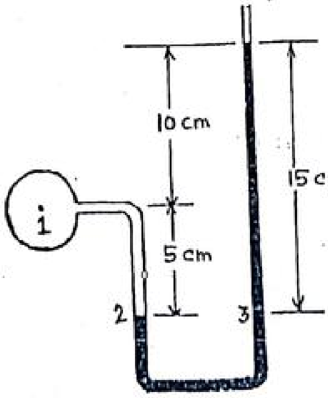
The following is a general procedure for solving of all manometer problems:

1. Start at one end (or any meniscus if the circuit is continuous) and write the pressure there in an appropriate unit or in an appropriate symbol if it is unknown.
2. Add to this the change in pressure, in the same unit, from one meniscus to the next (plus if the next meniscus is lower, minus if higher).
3. Continue until the other end of the gage (or the starting meniscus) is reached and equate the expression to the pressure at that point, known or unknown.

The expression will contain one unknown for a simple manometer or will give a difference in pressure for a differential manometer.

Example 2.3

A U-tube manometer containing Hg (sp.gr.13.6) has its right limb opened to the atmosphere as shown in the figure. The left limb is full of water and is connected to a pipe containing water under pressure. Find the pressure of water in the pipe above atmosphere for the manometer readings as shown in the figure.



Solution Equating the pressure at 2-3, we have

$$h_1 \times 1 + 0.05 \times 1 = 0.15 \times 13.6$$

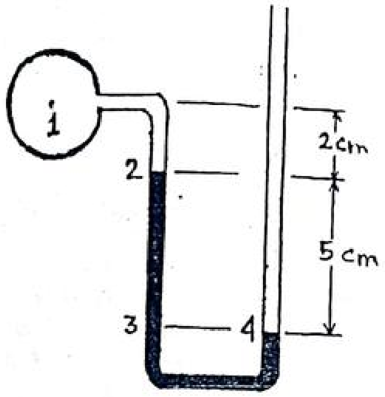
or,

$$h_1 = 0.15 \times 13.6 - 0.05 \times 1 = 1.99 \; m \; of \; water$$

$$\therefore p_1 = \gamma h_1 = 9.81 \times 1.99 = 19.52 \; kN/m^2$$

Example 2.4

A U-tube manometer containing Hg (sp.gr.13.6) is used to measure negative pressure in a pipe containing water as shown in the figure. The right limb of the manometer is open to the atmosphere. Find the negative pressure below the atmosphere in the pipe for the manometer readings as given in the figure.



Solution Equating the pressure at 3-4, we have

$$h_1 + 0.02 \times 1 + 0.05 \times 13.6 = 0$$

$$\therefore h_1 = -0.05 \times 13.6 - 0.02 \times 1 = -0.70 \; m \; of \; water$$

$$= 0.70 \text{ m of water (vacuum)}$$

$$\therefore p_1 = \gamma h_1 = -9.81 \times 0.70 = -6.867 \; kN/m^2$$

$$= 6.867 \text{ kN/m}^2 \text{ (vacuum)}$$

Differential Manometer: A differential manometer (Fig. 2.5) is used to determine the difference of pressure at two points A and B when the actual pressure at any point in the system can not be determined. The two points A and B may be in the same pipeline or in two different pipes.

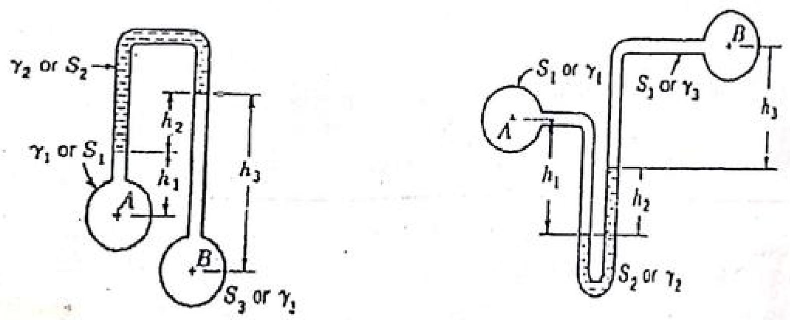


Fig. 2.5 Differential manometer

Application of the procedure outlined above to Fig. 2.5(a) produces

$$p_A - h_1\gamma_1 - h_2\gamma_2 + h_3\gamma_3 = p_B \qquad (2.15)$$

or

$$p_A - p_B = h_1\gamma_1 + h_2\gamma_2 - h_3\gamma_3 \qquad (2.16)$$

Similarly, for Fig. 2.5(b),

$$p_A + h_1\gamma_1 - h_2\gamma_2 - h_3\gamma_3 = p_B \quad \text{or} \quad p_A - p_B = -h_1\gamma_1 + h_2\gamma_2 + h_3\gamma_3 \qquad (2.17)$$

No formulas for particular manometers should be memorized. It is much more satisfactory to work them out from the general procedure for each case as needed.

If the pressures at A and B are expressed in length of the water column, the above results can be written, for Fig. 2.5(a), as

$$h_A - h_B = h_1S_1 + h_2S_2 - h_3S_3 \quad \text{units of length of water} \qquad (2.18)$$

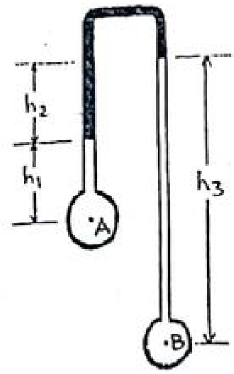
Similarly, for Fig. 2.5(b)

$$h_A - h_B = -h_1S_1 + h_2S_2 + h_3S_3 \qquad (2.19)$$

in which S_1, S_2 and S_3 are the specific gravities of the liquids in the system.

Example 2.5

In the figure shown, the liquids at A and B are water and the manometer liquid is oil (sp. gr. = 0.80). (a) If h_1 = 30 cm, h_2 = 20 cm and h_3 = 60 cm, determine $p_A - p_B$. (b) If p_B = 50 kN/m^2 and the barometric pressure is 76 cm of Hg (sp.gr.13.6), find the pressure at A in m of water absolute.



Solution

(a) $h_A - 0.3\times1 - 0.2\times0.8 + 0.6\times1 = h_B$

$\therefore\ h_A - h_B = 0.3\times1 + 0.2\times0.8 - 0.6\times1 = -0.14\ m\ of\ water$

$\therefore\ p_A - p_B = \gamma(h_A - h_B) = -9.81\times0.14 = -1.373\ kN/m^2 = -1373\ N/m^2$

(b) $h_B = \dfrac{p_B}{\gamma} = \dfrac{50}{9.81} = 5.097\ m\ of\ water$

h_B (abs) = h_B (gage) + 0.76 × 13.6 = 5.097 + 10.336 = 15.433 m of water

∴ From (a), h_A(abs) = h_B(abs) – 0.14 = 15.433 – 0.14 = 15.293 m of water

Manometer Liquid: For measuring high pressures or large pressure differences, a heavy liquid such as mercury (sp. gr. = 13.57) is employed. For small pressure differences, a light fluid, such as oil, or even air, may be used. Naturally, the liquid must be one that will not mix or react chemically with the fluid whose pressure is to be determined.

2.6 HYDROSTATIC PRESSURE FORCE ON SUBMERSED PLANE SURFACES

Total Pressure Force on Submerged Plane Surfaces

Consider a plane surface MN acted upon by a liquid of unit weight γ. The surface makes an angle θ with the horizontal (Fig. 2.6). Consider the surface MN to be made up of an infinite number of horizontal strips each having an area dA and a width dy so small that the unit (or intensity of) pressure on a strip may be considered constant. The unit pressure on a strip at depth h below the free surface and at distance y from the line S-S is

$$p = \gamma h = \gamma y \sin\theta \qquad (2.20)$$

The total pressure force on the strip is

$$dP = \gamma y \sin\theta\, dA \qquad (2.21)$$

and the total pressure force on MN is

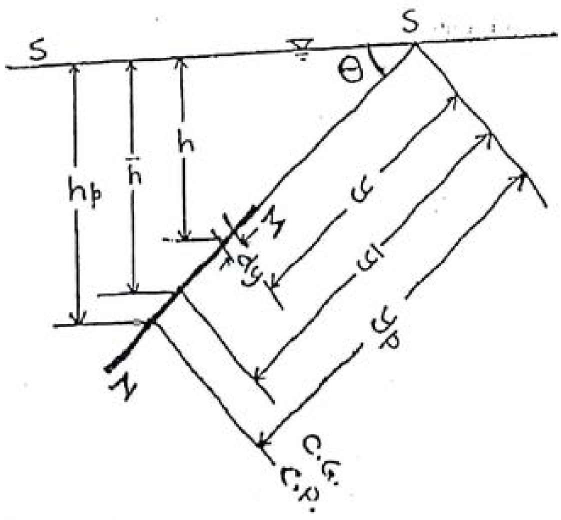


Fig. 2.6 Pressure on a plane surface

$$P = \gamma \sin\theta \int_0^A y\,dA \qquad (2.22)$$

From the definition of center of gravity

$$\int_0^A y\,dA = A\bar{y} \qquad (2.23)$$

where $\bar{y}$ is the distance of the center of gravity of A from the line S-S. Hence,

$$P = \gamma \sin\theta\, A\,\bar{y} \qquad (2.24)$$

Since the vertical depth of the center of gravity below the surface is

$$\bar{h} = \bar{y}\sin\theta \qquad (2.25)$$

it follows that

$$P = \gamma \bar{h} A \qquad (2.26)$$

where $\gamma \bar{h}$ represents the unit (or intensity of) pressure at the center of gravity of A.

Thus, **the total hydrostatic pressure force on any plane surface is equal to the product of the area of the surface and the unit pressure at its center of gravity.**

Total pressure force on a horizontal surface: The total pressure force on the horizontal surface (Fig. 2.7) is given by

$$P = \gamma \bar{h} A = \gamma h A \qquad (2.27)$$

where γ is the specific weight of the liquid (N/m^3), A is the area of the surface (m^2) and $\bar{h}$ = h is the depth of the horizontal surface from the free liquid surface (m).

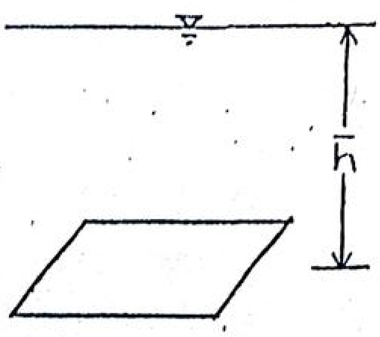


Fig. 2.7 Horizontal immerged surface

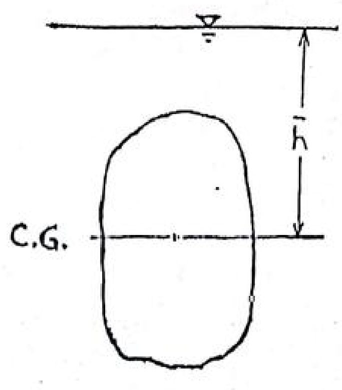


Fig. 2.8 Vertical immerged surface

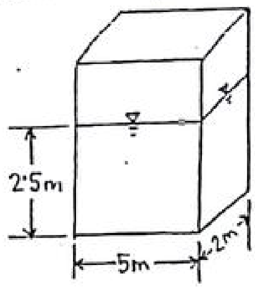
Total pressure force on a vertical surface: The total pressure force on the vertical surface is given by

$$P = \gamma \bar{h} A \qquad (2.28)$$

where γ is the specific weight of the liquid (N/m³). A is the area of the surface (m²) and $\bar{h}$ is the depth of the center of gravity of the immerged surface from the free liquid surface (m).

Example 2.6

A rectangular tank having a base 5 m long and 2 m wide contains water up to a depth of 2.5 m. Calculate the total pressure force on the base of the tank.



Solution Area of the base of the tank, $A = 5 \times 2 = 10\ m^2$

Depth of the base from the water surface, $\bar{h} = 2.5$ m

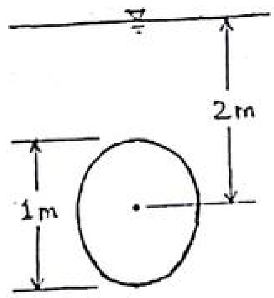
Specific weight for water, $\gamma = 9.81\ kN/m^3$

Hence, the total pressure force on the base

$$P = \gamma \bar{h} A = 9.81 \times 2.5 \times 10 = 245.25\ kN$$

Example 2.7

A circular door of 1 m diameter closes an opening in the vertical side of a bulkhead, which retains sea water. The center of the opening is at a depth of 2 m from the water level. Determine the total pressure force on the door. Take the specific gravity of sea water as 1.03.



Solution Area of the door, $A = \pi d^2/4 = \pi \times 1^2/4 = 0.785\ m^2$

Depth at the center of the door from the water surface, $\bar{h} = 2$ m

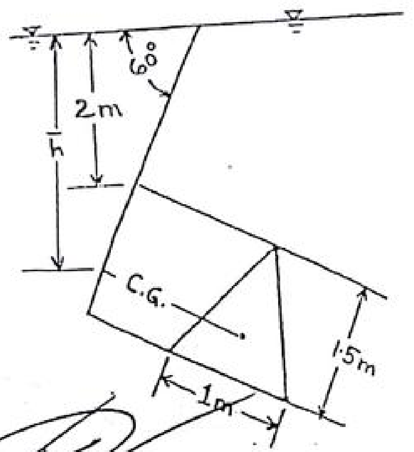
Specific weight for sea water, $\gamma = 9.81 \times 1.03 = 10.10\ kN/m^3$.

Hence, the total pressure force on the door

$$P = \gamma \bar{h} A = 10.10 \times 2 \times 0.785 = 15.865\ kN$$

Example 2.8

Find the total pressure force on a triangular plate whose base is 1 m and altitude is 1.5 m as shown in the figure. The plane of the plate is inclined at an angle of 60⁰ to the free surface of water.



Solution Vertical depth of C.G. of the plate from the water surface

$$\bar{h} = 2 + \frac{2}{3} \times 1.5 \times \sin 60^0 = 2 + 0.866 = 2.866\ m$$

$$\text{Area of the plate, } A = \frac{1}{2} \times 1 \times 1.5 = 0.75\ m^2$$

Specific weight for water, $\gamma = 9.81\ kN/m^3$

Hence, the total pressure force on the plate

$$P = \gamma \bar{h} A = 9.81 \times 2.866 \times 0.75 = 21.09\ kN$$

Center of Pressure of Submerged Plane Surfaces

The intensity of pressure on an immerged surface is not uniform, but increases with depth. As the pressure is greater over the lower part of the surface, the resultant pressure force will act at some point which is below the center of gravity of the immerged surface and towards the lower edge of the surface. The point through which the resultant pressure force acts is known as the center of pressure and is always expressed in terms of depth from the free liquid surface.

The position of the center of pressure of a plane surface subjected to hydrostatic pressure may be determined by taking moments of all the forces acting on the surface about some horizontal axis in its plane. In Fig. 2.6, the line S-S may be taken as the axis of moments for the surface MN. Designating by y_p the distance to the center of pressure from the axis of moments, it follows from the definition of the center of pressure that

$$P.y_p = \int y dP \tag{2.29}$$

or

$$y_p = \frac{\int y dP}{P} \tag{2.30}$$

But, as in the previous article

$$dP = \gamma y \sin\theta \, dA \tag{2.31}$$

and

$$P = \gamma \sin\theta \, A \bar{y} \tag{2.32}$$

Substituting Eq.(2.32) in Eq.(2.30), we obtain

$$y_p = \frac{\gamma \sin\theta \int y^2 dA}{\gamma \sin\theta \, A \bar{y}} = \frac{\int y^2 dA}{A \bar{y}} = \frac{I_S}{S_S} \tag{2.33}$$

where I_S is the moment of inertia or second moment of area of MN about the axis S-S from which y is measured and S_S is the static moment or first moment of area of MN about the same axis.

We know from the theory of parallel axis that

$$I_s = I_g + A\bar{y}^2 = I_g + \frac{A\bar{h}^2}{\sin^2\theta} \tag{2.34}$$

where I_g is the moment of inertia about an axis passing through its center of gravity and parallel to OS (Fig. 2.6), $\bar{y}$ = distance between the liquid surface and the center of gravity of the area along MN and $\bar{h}$ is the vertical depth of the center of gravity of the area from the free liquid surface. Obviously,

$$\bar{y} = \frac{\bar{h}}{\sin\theta} \tag{2.35}$$

If h_p is the vertical depth of the center of pressure of the area from the free liquid surface, then

$$h_p = y_p \sin\theta = \frac{I_S \sin\theta}{S_S} = \left(\frac{I_g + A\bar{y}^2}{A\bar{y}}\right)\sin\theta = \bar{y}\sin\theta + \frac{I_g}{A\bar{y}}\sin\theta \tag{2.36}$$

Since $\bar{h} = \bar{y}\sin\theta$, we obtain

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}}\sin^2\theta \tag{2.37}$$

The above relationships for h_p do not contain any term for specific weight of liquid. Therefore, the center of pressure will remain the same for all liquids.

In Eq.(2.37), if $\theta = 0$, then it becomes the case of horizontal plan surface and the center of gravity and the center of pressure coincide.

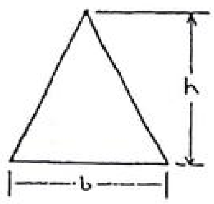
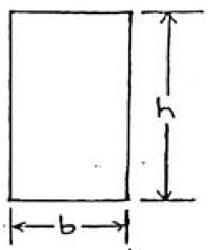
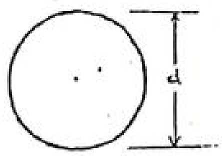
For vertical plane surfaces, $\theta = 90^0$, and Eq.(2.37) gives

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}} \tag{2.38}$$

Also, Eq. (2.37) indicates that $h_p > \bar{h}$, i.e. the center of pressure is always below the center of gravity of the area and

$$e = h_p - \bar{h} = \frac{I_g}{A\bar{h}}\sin^2\theta \qquad (2.39)$$

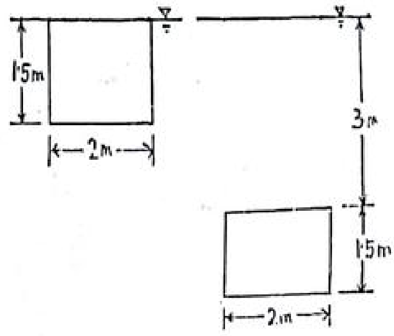
Table 2.1 The center of gravity (C.G.) and the moment of inertia (I) of some surfaces

Sl. No.	Surface	C.G. from base	I_g	I from base
1.Triangle		$\frac{h}{3}$	$\frac{bh^3}{36}$	$\frac{bh^3}{12}$
2.Rectangle		$\frac{h}{2}$	$\frac{bh^3}{12}$	$\frac{bh^3}{3}$
3.Circle		$\frac{d}{2}$	$\frac{\pi d^4}{64}$	

Example 2.9

A rectangular gate which is 2 m wide and 1.5 m high is in a vertical plane. The water surface coincides with the top of the gate. (a) Find the force exerted by the water on the gate and the position of the center of pressure. (b) If the same area is immerged vertically downwards, the top side being at a depth of 3 m below the free surface, find the force exerted by the water on the gate and the position of the center of pressure.

Solution



(a) Total pressure force, $P = \gamma \bar{h} A$

Here, $\gamma = 9.81$ kN/m^3

$\bar{h} = 1.5/2 = 0.75$ m

$A = 2 \times 1.5 = 3$ m^2

$\therefore P = \gamma \bar{h} A = 9.81 \times 0.75 \times 3 = 22.07\ kN$

This is the force exerted by the water on the gate.

Depth of center of pressure below the free water surface

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}} = 0.75 + \frac{bh^3/12}{A\bar{h}} = 0.75 + \frac{2\times1.5^3}{12\times3\times0.75} = 0.75 + 0.25 = 1\ m$$

(b) Total pressure force

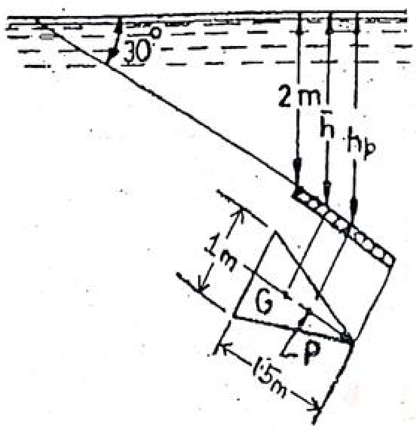
$$P = \gamma\bar{h}A = 9.81\times(3+0.75)\times3 = 110.36\ kN$$

and the position of the center of pressure below the free water surface is

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}} = 3.75 + \frac{bh^3/12}{A\bar{h}} = 3.75 + \frac{2\times1.5^3}{12\times3\times3.75} = 3.75 + 0.05 = 3.80\ m$$

Example 2.10

A triangular plate of 1 m base and 1.5 m altitude is immersed in water as shown in the figure. The plane of the plate is inclined at 30^0 with the free surface of water and the base is parallel to and at a depth of 2 m from the free surface. Find the total pressure force on the plate and the position of the center of pressure.



Solution Total pressure force, $P = \gamma\bar{h}A$

Here, $\gamma = 9.81\ kN/m^3$

$$\bar{h} = 2 + \frac{1.5}{3}\sin30^0 = 2 + 0.5\times\frac{1}{2} = 2.25\ m$$

$$A = \frac{1}{2}\times1\times1.5 = 0.75\ m^2$$

$$\therefore P = \gamma\bar{h}A = 9.81\times2.25\times0.75 = 16.55\ kN$$

The position of the center of pressure below the free water surface is given by

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}}\sin^2\theta = \bar{h} + \frac{bh^3/36}{A\bar{h}}\sin^2\theta = 2.25 + \frac{1\times1.5^3}{36\times0.75\times2.25}\times\sin^2 30^0$$

$$= 2.25 + 0.014 = 2.264\ m$$

2.7 FORCES ON SUBMERGED CURVED SURFACES

In case of a submerged plane surface, the forces acting perpendicularly on all the strips have the same directions and form a system of parallel forces. But in case of a submerged curved surface, all the strips do not lie in the same plane and the forces, though perpendicular to their respective strips, do not form a system of parallel forces. Due to this reason, the magnitude of the total force and its point of application (i.e. center of pressure) cannot be determined easily by the methods explained earlier. However, the same can be conveniently obtained by calculating the horizontal and vertical components of the resultant or total pressure force, which are then combined together to give the total force on the curved surface.

Consider a curved surface AB immersed in a liquid. Let BC be the vertical projection and AC be the horizontal projection of the curved surface (Fig. 2.9). The horizontal force P_H will be the total horizontal pressure force on the projection BC of the curved surface and will act through the center of pressure of the surface. The vertical force P_V will be the total weight of the liquid in the portion ABC and will act through the center of gravity of the volume ABC.

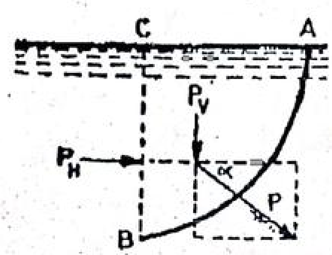


Fig. 2.9 Resultant pressure on a curved surface

The resultant of P_H and P_V will be the resultant or total force P acting on the curved surface, i.e.

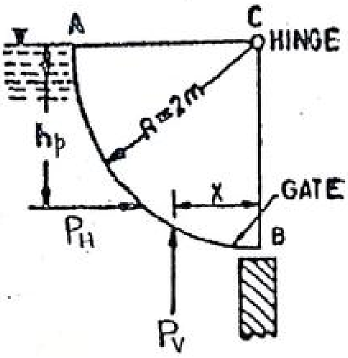
$$P = \sqrt{P_H^2 + P_V^2} \qquad (2.40)$$

The inclination of the resultant force with the horizontal α is given by

$$\alpha = \tan^{-1} \frac{P_V}{P_H} \qquad (2.41)$$

Example 2.11

(a) Find the total pressure force acting on the gate per m length, which is a quadrant of a circular cylinder of radius 2 m, as shown in the figure. (b) At what angle will it be acting to the horizontal? (c) Prove that the resultant force passes through the hinge C. (d) Give the reason why the resultant passes through the hinge C.



Solution (a) The horizontal force P_H due to water is the force on vertical projection of AB, acting at the center of pressure. Therefore,

$$P_H = \gamma \bar{h} A = 1000 \times \frac{2}{2} \times (2 \times 1) = 2000\ kg$$

per m length acting at depth

$$h_p = \bar{h} + \frac{I_g}{A\bar{h}} = \bar{h} + \frac{bh^3/12}{A\bar{h}} = 1 + \frac{1 \times 2^3/12}{(1 \times 2) \times 1} = 1.333\ m$$

from the free surface.

The vertical component of the resultant pressure force acting on the curved surface is

$$P_V = \gamma V = 1000 \times (\frac{\pi \times 2^2}{4} \times 1) = 3140\ kg$$

per m length acting vertically upward through the center of gravity of the volume of water. Now, the center of gravity of a quadrant of a circle is located at a distance of $4R/3\pi$ from B.

$$\therefore\ x = \frac{4R}{3\pi} = \frac{4 \times 2}{3\pi} = 0.849\ m$$

from B as shown in the figure.

Therefore, the resultant force acting on the gate

$$P = \sqrt{P_H^2 + P_V^2} = \sqrt{2000^2 + 3140^2} = 3720\ kg$$

per m length.

(b) The angle which the resultant force makes with the horizontal is

$$\alpha = \tan^{-1} \frac{P_V}{P_H} = \tan^{-1} \frac{3140}{2000} = \tan^{-1} 1.57 = 57.5^0$$

(c) The sum of moments of the forces P_H and P_V about C is

$$P_H \times h_p - P_V \times x = 2000 \times 1.333 - 3140 \times 0.849 = 0$$

Therefore, the resultant force passes through C.

(d) Since the pressure forces at all points act perpendicular to the circular gate AB, so the line of action of their resultant pass through C.

2.8 PRACTICAL APPLICATION OF HYDROSTATICS

In the field of engineering, the principle of hydrostatics is utilized to determine the hydrostatic pressure exerted on a hydraulic structure. The hydraulic structures are sluice gates, weirs, lock gates, masonry walls, dams, etc. Thus, the study of the subject hydrostatics is of much importance while designing all types of hydraulic structures.

PROBLEMS AND EXERCISES

2.1 State Pascal's law.

2.2 Derive the expression for pressure in a static fluid and show that the pressure is independent of x and z, and depends on y only.

2.3(a) Define absolute and gage pressures. What is the relationship between them?
(b) In a neat sketch, show the units and scales for pressure measurement.

2.4(a) Derive the expression for (i) the total pressure force on a submersed plane surface, and (ii) the center of pressure of a submerged plane surface.
(b) Explain why the center of pressure is always below the center of gravity of a submerged area.

2.5 State the use of (i) manometer, (ii) piezometer, (iii) differential manometer, (iv) mercury barometer, (v) Burdon pressure gage, and (vi) Aneroid barometer.

2.6 State the name of a manometer liquid. What is the criterion for selecting a manometer liquid?

2.7 Determine the depth of water which will produce a pressure intensity of 100 kN/m^2.

2.8 What is the pressure exerted in kN/m^2 at a point 1 m below the free surface of water (or by a 1 m vertical column of water)?

2.9 The atmospheric pressure is 760 mm of Hg (sp. gr. = 13.6). Convert this pressure to (i) equivalent depth of water, and (ii) kN/m^2.

2.10 A cylindrical water tank 10 m in diameter and 15 m high is filled with water. Determine (i) the intensity of pressure at the bottom of the tank, (ii) the total pressure force on the bottom, (iii) the maximum, the minimum and the average intensities of pressure on the vertical wall, and (iv) the total pressure force on the vertical wall.

2.11 A U-tube manometer is connected to a conical vessel as shown in Fig. 2.10. The reading of the manometer when the vessel is empty is shown in the figure. Find the reading of the manometer when the vessel is completely filled with water.

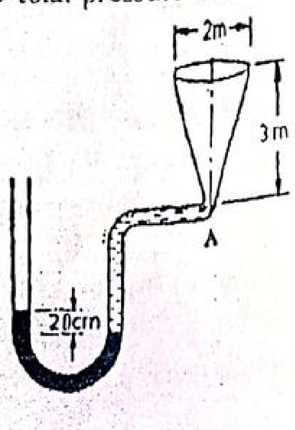


Fig. 2.10 (Problem 2.11)

2.12 A trapezoidal surface shown in Fig. 2.11 is immersed in a liquid of specific gravity 0.8 at an inclination of 30^0 to the free surface. Find the hydrostatic force on it.

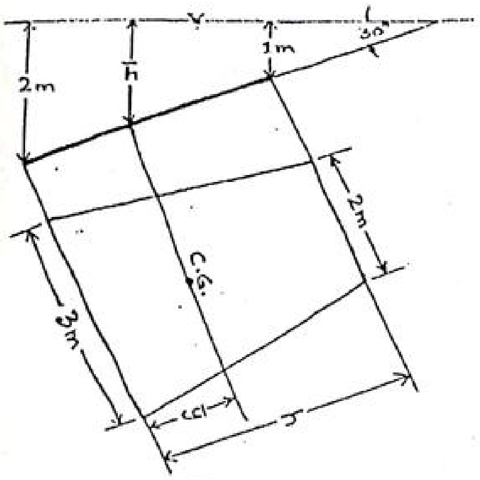


Fig. 2.11 (Problem 2.12)

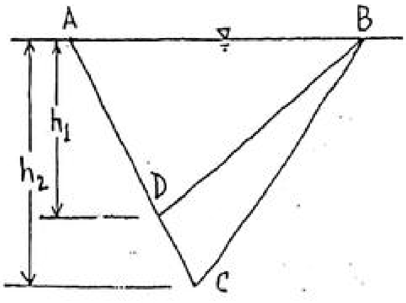


Fig. 2.12 (Problem 2.13)

2.13 A vertical triangular lamina is immersed in water with the side AB coinciding the water surface as shown in Fig. 2.12. A point D is taken in AC such that the pressure forces on the two areas ABD and DBC are equal. Show that $AD:AC = 1:\sqrt{2}$.

2.14 A rectangular gate 6 m×2 m is hinged at its base and inclined at 60^0 to the horizontal as shown in Fig. 2.13. The upper end of the gate is kept in position by a weight W equal to 5500 kg attached as shown. Find the level of water, when the gate begins to fall, neglecting the weight of the gate and friction at the horizontal pulley.

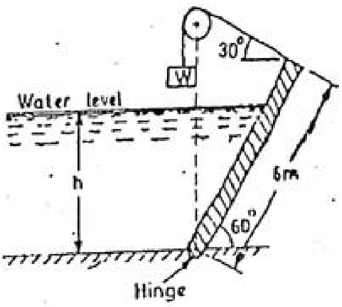


Fig. 2.13 (Problem 2.14)

2.15 Determine the total pressure force acting on the curved gate AB, per m length, which is the quadrant of a circular cylinder of radius 1 m as shown in Fig. 2.14. Also, determine the angle which the total pressure force makes with the horizontal.

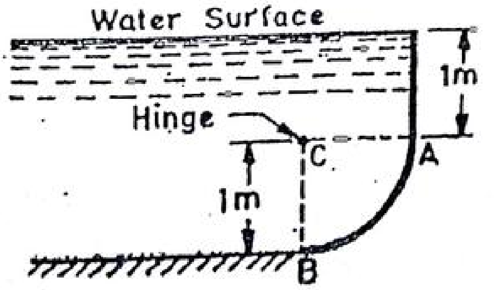


Fig. 2.14 (Problem 2.15)

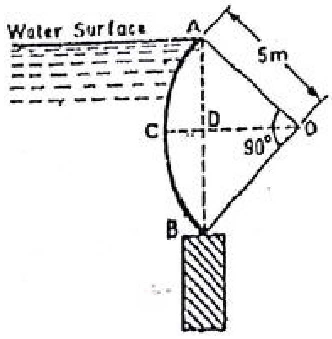


Fig. 2.15 (Problem 2.16)

2.16 A tainter gate is subjected to water pressure as shown in Fig. 2.15. Determine the horizontal and vertical pressure forces acting on the face of the gate.

2.17 A roller gate of cylindrical form 3 m in diameter has a span of 10 m (Fig. 2.16). Find the magnitude and direction of the resultant pressure force acting on the cylinder, when it is placed on the dam and the water level is such that it is going to spill.

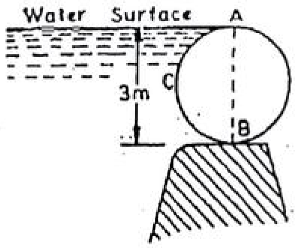


Fig. 2.16 (Problem 2.17)

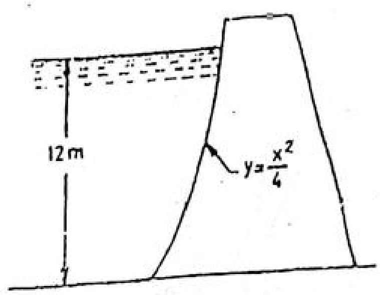


Fig. 2.17 (Problem 2.18)

2.18 The curved face of a dam, retaining water, is shaped according to the relationship $y = x^2/4$, as shown in Fig. 2.17. The height of water retained by the dam is 12 m. Find the magnitude and direction of the resultant force due to water pressure on the dam.

****

Chapter 3

BUOYANCY AND FLOATATION

3.1 INTRODUCTION

A body immersed partially or fully in a fluid experiences a vertical upward force, known as the *buoyant force*. The tendency of a submerged body to rise in a fluid because of the buoyant force, which opposes the downward force of gravity or weight of the fluid, is known as *buoyancy*. The magnitude of the buoyant force can be determined by the Archimedes' principle.

3.2 ARCHIMEDES' PRINCIPLE

The Archimedes' principle states that *when a body is immersed wholly or partly in a fluid, it is buoyed (or lifted) up by a force equal to the weight of the fluid displaced by the body.*

Proof: Let a body be immersed in a fluid of constant specific weight γ. Divide the body into a large number of vertical prisms (Fig. 3.1). Consider an elementary vertical prism of cross-sectional area dA.

Force acting upward on the bottom of the prism

$$= p_2 dA = \gamma h_2 dA \qquad (3.1)$$

Force acting downward on the top of the prism

$$= p_1 dA = \gamma h_1 dA \qquad (3.2)$$

∴ Net upward force on the prism

$$dP_B = \gamma (h_2 - h_1) dA = \gamma dV \qquad (3.3)$$

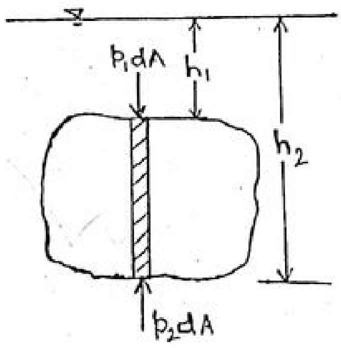


Fig. 3.1 Archimedes' principle

where dV is the volume of the prism.

∴ Total upward force or buoyant force

$$P_B = \gamma \int dV = \gamma V \qquad (3.4)$$

where V is the total volume of the body. Obviously, γV is the weight of the fluid displaced by the body. Hence, it is proved that the buoyant force is equal to the weight of fluid displaced.

The buoyant force is vertical and acts through the center of gravity of the displaced fluid. The point of application of the buoyant force is known as the *center of buoyancy.*

A body can be made to float (i) by decreasing the weight of the body, the volume remaining the same (e.g. a submarine), and (ii) by increasing the volume of the body, weight remaining the same (e.g. a ship).

Example 3.1

A block of wood 4 m long, 2 m wide and 1 m deep is floating horizontally in water. If the density of wood is 700 kg/m^3, find the volume of water displaced and the position of the center of buoyancy.

Solution Volume of block = = $4 \times 2 \times 1 = 8\ m^3$

Weight of block = $8 \times 700 = 5600$ kg

Now, since the block of wood floats in water, so

Weight of block = weight of water displaced

= density of water × volume of water displaced

$$\therefore \text{Volume of water displaced} = \frac{Weight\ of\ block}{Density\ of\ water} = \frac{5600}{1000} = 5.6\ m^3$$

$$\text{Depth of immersion of the block} = \frac{Volume\ of\ water\ displaced}{Sectional\ area\ of\ block} = \frac{5.6}{4 \times 2} = 0.7\ m$$

$$\therefore \text{Center of buoyancy} = \frac{0.7}{2} = 0.35 \text{ m from base}$$

Example 3.2

A submarine has a total enclosed volume of $1000\ m^3$. If it weighs 500 tons, calculate the volume of water to be pumped into the submarine in order that it is submerged.

Solution $V = 1000\ m^3$, $W = 500$ tons $= 500 \times 1000$ kg $= 500 \times 1000 \times 9.81$ N $=$ 500×9.81 kN= 4905 kN ($\therefore$ 1 ton = 1000 kg)

Upward or buoyant force on the submarine = weight of water displaced

or

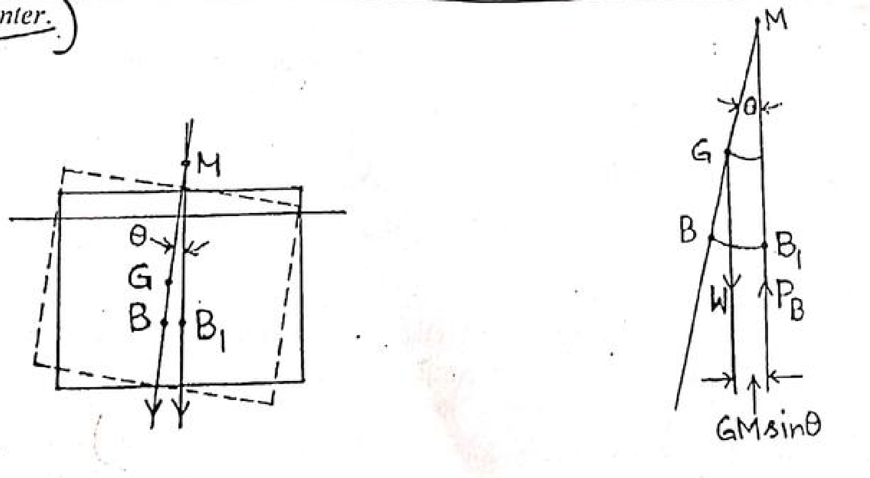
$$P_B = \gamma \times V = 9.81 \times 1000 = 9810 \text{ kN}$$

Since $W < P_B$, the submarine is floating. The minimum volume of water to be pumped so that it may result in sinking of the submarine

$$= \frac{9810 - 4905}{9.81} = 500\ m^3$$

3.3 METACENTRIC HEIGHT

Whenever a body, floating in a liquid, is given a small angular displacement, it starts oscillating about some point. This point, about which the body starts oscillating, is called the *metacenter.*



(a) (b)

Fig. 3.2 Metacenter

When the floating body is at rest, both the center of gravity G and the center of buoyancy B lie in the same vertical line (Fig. 3.2). Also, W and P_B act through the same vertical line, but in opposite direction. If the body is slightly tilted by a small angle of heel θ, the center of gravity remains the same, but the center of buoyancy B will change to B_1. The point of intersection M of the original vertical line passing through B and G and the vertical line passing through the new center of buoyancy B_1 is called the metacenter. The distance between the center of gravity G of the floating body and the metacenter M is called the *metacentric height.*

The metacentric height of a floating body is a direct measure of its stability. If the metacentric height is more, the body will be more stable. The metacentric height of different ships may vary from 0.3 m to 3.6 m.

3.4 DETERMINATION OF METACENTRIC HEIGHT AND RIGHTING MOMENT

Consider a ship floating in water. Let the ship be given a clockwise rotation through a small angle θ (Fig. 3.3). The immersed section has now changed from acde to acd_1e_1.

The original center of buoyancy B has now changed to a new position B_1. The triangular wedge Oam has come out of water, whereas the triangular wedge Ocn has gone under water. Since the volume of water displaced remains the same, so the two triangular wedges must have the equal areas.

The triangular wedge Oam has come out of water, thus decreasing the force of buoyancy on the left, therefore it tends to rotate the ship in an anticlockwise direction. Similarly, as the triangular wedge Ocn has gone under water, thus increasing the force of buoyancy on the right, therefore it again tends to rotate the ship in an anticlockwise direction. So, these two forces of buoyancy will form a couple which will tend to rotate the ship in an anticlockwise direction about O. If θ is very small, then the ship may be assumed to rotate about the metacenter M.

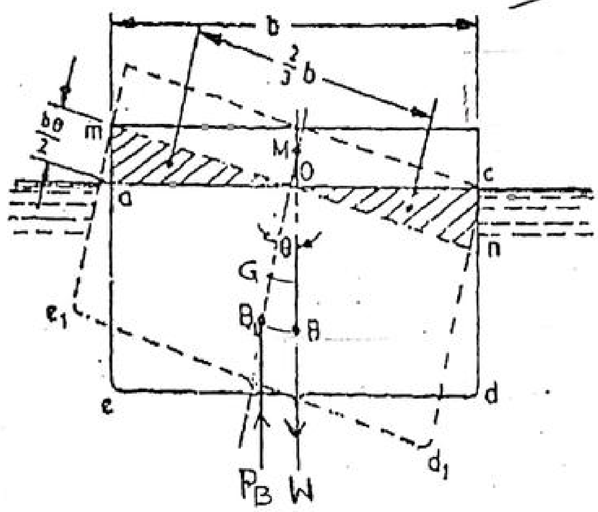


Fig. 3.3 Metacentric height

Let l be the length of the ship, b be the width of the ship and V be the volume of water displaced by the ship. From the geometry of the figure, we find that

$$am = cn = \frac{b\theta}{2} \tag{3.5}$$

where θ is in radians.

$$\therefore \text{Volume of wedge of water Oam} = \frac{1}{2} \times \frac{b}{2} \times \frac{b\theta}{2} \times l = \frac{b^2\theta l}{8} \tag{3.6}$$

$$\therefore \text{Weight of this wedge of water} = \frac{\gamma b^2\theta l}{8} \tag{3.7}$$

where γ is the specific weight of water.

The arm of the couple $= \frac{2}{3}b$. Therefore,

$$\text{Moment of the rotating couple} = \frac{\gamma b^2\theta l}{8} \times \frac{2}{3}b = \frac{\gamma b^3\theta l}{12} \tag{3.8}$$

and

$$\text{Moment of the disturbing force} = P_B \times BB_1 = \gamma V \times BB_1 \tag{3.9}$$

Equating these two moments, we obtain

$$\frac{\gamma b^3 \theta l}{12} = \gamma \times V \times BB_1 \qquad (3.10)$$

Substituting $lb^3/12 = I$ and $BB_1 = BM \times \theta$ in the above equation, we get

$$\gamma I\theta = \gamma \times V \times BM \times \theta$$

$$\therefore BM = \frac{I}{V} \qquad (3.11)$$

where I is the moment of inertia of the plan of the ship about the longitudinal axis passing through O.

Now, the metacentric height (Fig. 3.2b) is

$$GM = BM \pm BG \qquad (3.12)$$

where the + sign is to be used if G is below B and – sign is to be used if G is above B.

Equation (3.11) is strictly valid for small angles of heel θ, i.e. when θ is up to 10^0. The righting moment is (Fig. 3.2b)

$$\text{R. M.} = W.\,GM.\sin\theta = \gamma.V.GM.\theta \qquad (3.13)$$

where θ is in radians.

Example 3.3

A ship is 20 m long, 10 m wide and 4 m deep. It has a draft of 2.5 m when floating in an upright position. The center of gravity of the ship is on the axis of symmetry, 0.5 m above the water surface. Compute (i) the metacentric height, and (ii) the righting moment, when the angle of heel is 10^0.

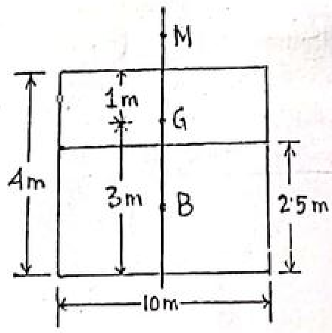
Solution

$$I = \frac{lb^3}{12} = \frac{20 \times 10^3}{12} = 1666.67\ m^4$$

$$V = 20 \times 10 \times 2.5 = 500\,m^3$$

$$BM = \frac{I}{V} = \frac{1666.67}{500} = 3.33\ m$$

$$BG = 0.5 \times 2.5 + 0.5 = 1.25 + 0.5 = 1.75\ m$$



Since BM > BG, so G is above B and

$$GM = BM - BG = 3.33 - 1.75 = 1.58\ m$$

$$\text{R. M.} = \gamma.V.GM.\theta = 1000 \times 9.81 \times 500 \times 1.58 \times \frac{10\pi}{180} = 1352612.72\ \text{N-m}$$

3.5 CONDITION OF EQUILIBRIUM OF A FLOATING BODY

By equilibrium (or stability) of a submerged or a floating body is meant its tendency either to return to or go away from its original position, when slightly disturbed. Therefore, a body, when slightly disturbed, may or may not return to its original position.

A submerged or floating body can have three possible conditions of equilibrium (or stability): stable, unstable and neutral.

Stable equilibrium: A body is said to be in a stable equilibrium, if it returns back to its original position, when it is given a small angular displacement. This happens when the metacenter (M) is higher than the center of gravity (G) of the floating body.

Unstable equilibrium: A body is said to be in an unstable equilibrium, if it does not return back to its original position, when it is given a small angular displacement. This happens when the metacenter (M) is lower than the center of gravity (G) of the floating body.

Neutral equilibrium: A body is said to be in a neutral equilibrium, if it occupies a new position and remains at rest in the new position, when it is given a small angular displacement. This happens when the metacenter (M) coincides with the center of gravity (G) of the floating body.

Example 3.4

A solid cylinder of 3 m diameter has a height of 3 m. It is made up of a material whose specific gravity is 0.8 and is floating in water with its axis vertical. Find its metacentric height and state whether its equilibrium is stable, unstable or neutral.

Solution Depth of immersion of the cylinder $= 0.8 \times 3 = 2.4$ m

$\therefore$ Distance of the center of buoyancy from the bottom of the cylinder

$$OB = \frac{2.4}{2} = 1.2\ m$$

and the distance of the center of gravity from the bottom of the cylinder

$$OG = \frac{3}{2} = 1.5\ m$$

$\therefore BG = OG - OB = 1.5 - 1.2 = 0.3$ m

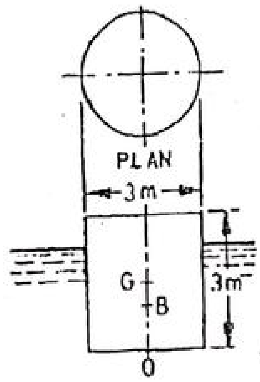
Now, the moment of inertia of the circular section

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 3^4}{64} = \frac{81\pi}{64}\ m^4$$

and the volume of water displaced

$$V = \frac{\pi}{4} \times 3^2 \times 2.4 = 5.4\pi\ m^3$$

$$\therefore BM = \frac{I}{V} = \frac{81\pi}{64 \times 5.4\pi} = 0.234\ m$$



So, the metacentric height

$GM = BM - BG = 0.234 - 0.3 = -0.066$ m

The $-$ sign indicates that the metacenter (M) is below the center of gravity (G). Therefore, the cylinder is in unstable equilibrium.

PROBLEMS AND EXERCISES

3.1 State Archimedes' principle.

3.2 Define metacenter and metacentric height.

3.3 Derive the equation

$$BM = I/V$$

3.4 Define the three types of equilibrium or stability and state the conditions for them.

3.5 A rectangular block of 100 cm × 50 cm × 50 cm floats in water with 1/9 of its volume being out of water. Find the weight of block.

3.6 A wooden block 4 m × 1 m × 0.5 m and of specific gravity 0.75 is floating in water. Determine the volume of concrete of specific weight 2500 kg/m^3, that may be placed on the block which will immerse (i) the block completely in water, and (ii) the block and the concrete completely in water.

3.7 A solid cylinder 36 cm long and of 8 cm diameter has a base 1 cm thick and of specific gravity 0.6 as shown in Fig. 3.4. The remaining part of the cylinder is of specific gravity 0.7. Determine if it can float vertically in water.

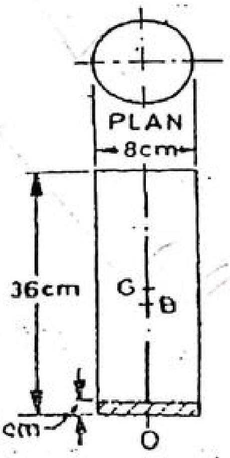


Fig. 3.4 (Problem 3.7)

3.8 A rectangular pontoon is 7 m long and 3 m wide. The weight of the pontoon is 30 tons. Determine the position of the center of gravity above the base of the pontoon such that it does not overturn in still water. Take the specific volume of water as 0.1 m^3/kN.

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Chapter 4

RELATIVE EQUILIBRIUM OF LIQUIDS

4.1 INTRODUCTION

In the previous chapters, liquids have been assumed to be in equilibrium and at rest both with respect to the earth and with respect to the containing vessel. In this chapter, we consider the condition when every particle of a liquid is at rest with respect to every other particle and with respect to the containing vessel, but the whole mass, including the vessel, has a uniformly accelerated motion with respect to the earth. The liquid is then in equilibrium and at rest with respect to the vessel, but it is neither in equilibrium not at rest with respect to the earth. In this condition, a liquid is said to be in *relative equilibrium*. Since there is no motion of the liquid with respect to the vessel and no movement between the fluid particles themselves, there can be no friction.

4.2 VESSEL MOVING WITH CONSTANT LINEAR ACCELERATION

Vessel Moving Horizontally

If a vessel partly filled with any liquid moves horizontally along a straight line with a constant acceleration a, the liquid surface will assume an angle θ with the horizontal (Fig. 4.1). To determine the value of θ, consider the forces acting on a small mass of liquid M at any point O on the surface. This mass is moving with a constant horizontal acceleration a and the force P, producing the acceleration, is the resultant of all the other forces acting upon the mass. These forces are the force of gravity W, acting vertically downward, and the pressure of all the contiguous particles of the liquid. The resultant P of all the pressure produced by these particles of liquid must be normal to the free surface AB. Since force equals mass times acceleration,

$$P = Ma = \frac{Wa}{g} \qquad (4.1)$$

and from Fig. 4.1

$$P = W \tan\theta \qquad (4.2)$$

From these two equations, we obtain

$$\tan\theta = \frac{a}{g} \qquad (4.3)$$

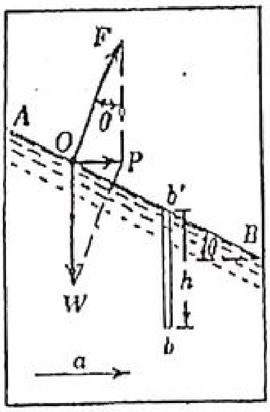


Fig. 4.1 Vessel with horizontal acceleration

Since O was assumed to be anywhere on the surface and the values of a and g are the same for all points, it follows that tan θ is constant at all points on the surface, or, in other words, AB is a straight line.

The same value of θ will hold for a vessel moving to the right with a positive acceleration as for a vessel moving to the left with a negative acceleration or a retardation.

To determine the intensity of pressure at any point b at a depth h below the free surface, consider the vertical forces acting on a vertical prism *bb'* (Fig. 4.1). Since there is no acceleration vertically, the only vertical forces acting are atmospheric pressure at *b'*, gravity and the upward pressure on the base of the prism at b. Hence, if the cross-sectional area is dA, then

$$p_b dA = \gamma h dA + p_a dA$$

or

$$p_b = \gamma h + p_a \qquad (4.4)$$

or, neglecting the atmospheric pressure which acts throughout

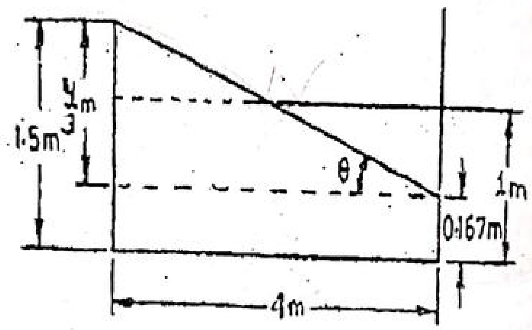
$$p_b = \gamma h \qquad (4.5)$$

Therefore, in a body of liquid moving with a horizontal acceleration, the relative pressure at any point is that caused by the head of liquid directly over the point, as in hydrostatics. In this case, however, all points of equal pressure lie in an inclined plane parallel to the surface of the liquid.

In Eq. (4.3), if $a = 0$, $\tan\theta = 0$. In other words, if a vessel moved horizontally with a constant velocity, the surface of the liquid would be horizontal.

Example 4.1

An open rectangular tank 4 m long, 2 m wide and 1.5 m deep contains water up to a depth of 1 m. (a) If the tank is moved horizontally parallel to its length with an acceleration 3.27 m/s², calculate the volume of water spilled out during the motion. (b) Also, calculate the forces acting on each end of the tank and the accelerating force on the fluid motion.



Solution (a) Volume of water in the tank before it is accelerated $= 4\times2\times1 = 8\ m^3$

Slope of the liquid surface after it is given constant acceleration

$$\tan\theta = \frac{a}{g} = \frac{3.27}{9.81} = \frac{1}{3}$$

∴ Rise of water level on the end of the tank = slope × half length $= \frac{1}{3}\times\frac{4}{2} = \frac{2}{3}$ m

∴ Water level on the rear end = initial water level + rise $= 1 + \frac{2}{3} = 1.667$ m

This level is greater than the height of the tank, i.e. 1.667 m > 1.5 m, so a volume of water will spill out of the tank. Now, as water will be contained in the tank up to a height of 1.5 m on rear end and the slope of the water surface will remain the same, i.e. $\tan\theta = 1/3$, so

$$\text{Depth of water at front end} = 1.5 - \frac{1}{3}\times 4 = 0.167\ m$$

∴ Volume of water retained in tank $= \left(\frac{1.5 + 0.167}{2}\right)\times 4\times 2 = 6.667\ m^3/s$

∴ Volume of water spilled out during the motion $= 8 - 6.667 = 1.333\ m^3/s$

(b) Force on the rear end face, $F_1 = \gamma \bar{h} A = 9.81\times\frac{1.5}{2}\times 1.5\times 2 = 22.07$ kN

Force on the front end face, $F_2 = \gamma \bar{h} A = 9.81 \times \frac{0.167}{2} \times 0.167 \times 2 = 0.27$ kN

$\therefore$ Accelerating force = = $F_1 - F_2 = 22.07 - 0.27 = 21.80$ kN

Alternatively,

Mass of water remained in the tank = 6.667 × 1000 = 6667 kg

Accelerating force on the fluid mass = M.a = 6667×3.27 = 21800 N = 21.80 kN

Vessel Moving Upward

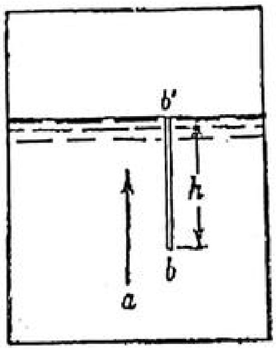
Consider the forces acting on a vertical prism of liquid bb' of height h and cross-sectional area dA (Fig. 4.2). The force P, producing the acceleration, is the resultant of all the forces acting on the prism, consisting of gravity equal to γhdA, acting downward, and the pressure on the lower end of the filament at b, equal to p_bdA, acting upward. Therefore,

$$P = p_b dA - \gamma\, hdA = Ma = \frac{\gamma\, hdA}{g} a$$

from which

$$p_b = \gamma h + \gamma h \frac{a}{g} = \gamma h(1 + \frac{a}{g}) \qquad (4.6)$$

This shows that the intensity of pressure at any point within a liquid contained in a vessel having an upward acceleration a is greater than the static pressure by an amount equal to γha/g. Evidently, if the acceleration were downward, the sign of the last term in the above expression would become negative, i.e.



$$p_b = \gamma h - \gamma h \frac{a}{g} = \gamma h(1 - \frac{a}{g}) \qquad (4.7)$$

Fig. 4.2 Vessel with vertical acceleration

Now, if a = g, p_b = 0. So, if a vessel containing any liquid falls freely, the pressure will be zero at all points throughout the vessel.

Example 4.2

An open rectangular tank 8 m long and 5 m wide contains an oil of specific gravity 0.90 up to a depth of 2.5 m. Determine the total pressure force on the bottom of the tank, when the tank is moving with an acceleration 4 m/s^2, (a) vertically upward, and (b) vertically downward.

Solution $\gamma = 0.90 \times 9.81 = 8.829$ kN/m^3

(a) When the tank is moving upward

Pressure at any depth h from the surface of the liquid

$$p = \gamma h(1 + \frac{a}{g}) = 8.829 \times 2.5 \times (1 + \frac{4}{9.81}) = 31.0725\ kN/m^2$$

Area of the bottom of tank $= 8 \times 5 = 40\ m^2$

$\therefore$ Total pressure force on the bottom $= 31.0725 \times 40 = 1242.9$ kN

(b) When the tank is moving downward

Pressure at any depth h from the surface of the liquid

$$p = \gamma h(1 - \frac{a}{g}) = 8.829 \times 2.5 \times (1 - \frac{4}{9.81}) = 13.073\ kN/m^2$$

$\therefore$ Total pressure force on the bottom $= 13.073 \times 40 = 522.90$ kN

4.3 VESSEL ROTATING ABOUT A VERTICAL AXIS

When the vessel shown in Fig. 4.3 is at rest, the surface of the liquid is horizontal and at mn. The form of the surface resulting from rotating the vessel with a constant angular velocity ω radians per second about its vertical axis OY is represented by $m'b'n'$. Consider the forces acting on a small mass of liquid M, at a, at a distance r from the axis OY.

Since this mass has a uniform circular motion, it is subjected to a centripetal force, $C = M\omega^2 r$, which produces an acceleration directed toward the center of rotation and is the resultant of all the other forces acting on the mass. These other forces are the force of gravity, $W = Mg$, acting vertically downward, and the pressure exerted by the adjacent particles of the liquid. The resultant force F of this liquid pressure must be normal to the free surface of the liquid at a.

Designating by θ the angle between the tangent at a and the horizontal, we have

$$\tan\theta = \frac{dh}{dr} = \frac{C}{W} = \frac{M\omega^2 r}{Mg} = \frac{\omega^2 r}{g}$$

or

$$dh = \frac{\omega^2 r}{g} dr \qquad (4.8)$$

which, when integrated, becomes

$$h = \frac{\omega^2 r^2}{2g} \qquad (4.9)$$

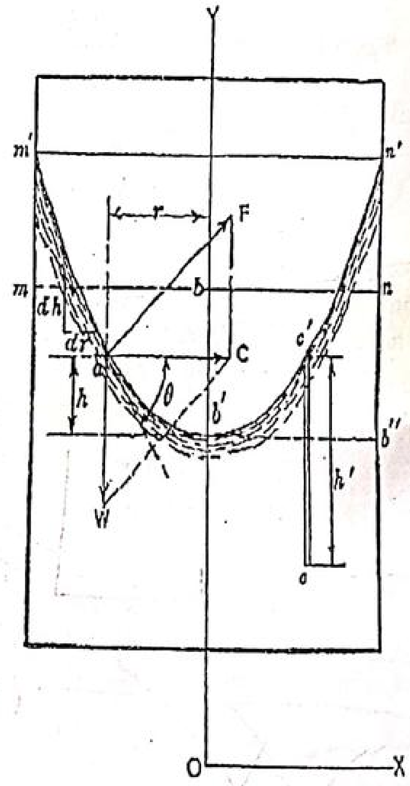


Fig. 4.3 Vessel rotating about a vertical axis

The constant of integration equals zero, because when r equals zero h also equals zero.

Since h and r are the only variables, Eq.(4.9) is an equation of parabola and the liquid surface is a paraboloid of revolution about the Y axis. As the volume of a paraboloid is equal to one-half that of the circumscribed circle and since the volume of liquid within the vessel has not changed,

$$b'b = \frac{1}{2} b''n' = nn'$$

The linear velocity at a is v = ωr. Substituting v for ωr in Eq.(4.9), we obtain

$$h = \frac{v^2}{2g} \qquad (4.10)$$

which means that, any point on the surface of the liquid will rise above the vertex of the paraboloid a height equal to the velocity head at that point, and this head is known as the centrifugal head.

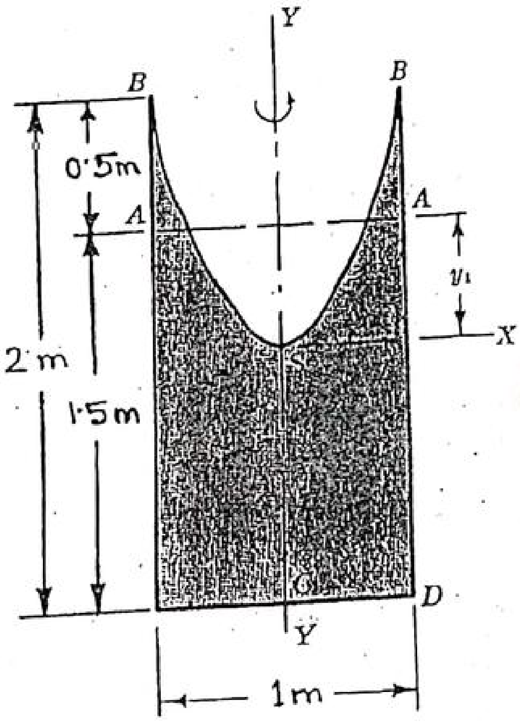
To determine the relative pressure at any point c at a depth h' vertically below the surface at c', consider the vertical forces acting on the prism cc' having a cross-sectional area dA. As this prism has no vertical acceleration, $\sum y = 0$ and $p_c dA = \gamma h' dA$. Hence,

$$p_c = \gamma h' \qquad (4.11)$$

That is, the relative pressure at any point is that caused by the head of liquid directly over the point, as in hydrostatics. Therefore, the distribution of pressure on the bottom of the vessel is represented graphically by the vertical ordinates to the curve $m'b'n'$. It also follows that the total pressure on the sides of the vessel is the same as though the vessel were filled to the level $m'n'$ and were not rotating.

Example 4.3

An open cylindrical tank 2 m high and 1 m in diameter contains 1.50 m of water. (a) What constant angular velocity can be attained without spilling any water? (b) What are the pressures at the bottom of the tank at the center and at the walls?



Solution Volume of paraboloid of revolution

$$= \frac{1}{2} \times \text{volume of circumscribed cylinder}$$

$$= \frac{1}{2} \times \frac{\pi}{4} \times 1^2 \times (0.50 + y_1)$$

If no liquid is spilled, this volume equals the volume above the original water level A-A,

or

$$\frac{1}{2} \times \frac{\pi}{4} \times 1^2 \times (0.50 + y_1) = \frac{\pi}{4} \times 1^2 \times 0.5$$

which gives

$$y_1 = 0.5 \text{ m}$$

Thus, the point on the axis of rotation drops by an amount equal to the rise of the liquid at the walls of the vessel.

From this information, the x and y coordinates of point B are respectively 0.5 m and 1 m from the origin. Then

$$y = \frac{\omega^2}{2g} x^2$$

or

$$1 = \frac{\omega^2}{2 \times 9.81} \times 0.5^2$$

or

$$\omega^2 = \frac{2 \times 9.81}{0.5^2} = 78.48$$

$$\therefore \omega = 8.86 \text{ rad/s}$$

The depth at the center of the tank C is 1 m and at the walls D the depth is 2 m.

$$\therefore \text{At C, } p = \gamma h = 9.81 \times 1 = 9.81 \text{ kN/m}^2$$
$$\therefore \text{At D, } p = \gamma h = 9.81 \times 2 = 19.62 \text{ kN/m}^2$$

PROBLEMS AND EXERCISES

4.1 What is meant by relative equilibrium ?

4.2 Show that when a vessel moves horizontally with a constant acceleration, the water surface assumes a straight line, and the points of equal pressure lie on an inclined plane parallel to the surface of the liquid.

4.3 Show that if a vessel containing any liquid falls freely, the pressure will be zero at all points throughout the vessel.

4.4 Show that when a vessel containing a liquid rotates about a vertical axis, the liquid surface is a paraboloid of revolution, and the point on the axis of rotation drops by an amount equal to the rise of liquid at the walls of the vessel.

4.5 An open rectangular tank 3 m long, 2.5 m wide and 1.5 m deep is completely filled with water. (a) If the tank is moved horizontally with an acceleration of 1.5 m/s^2, how many liters of water will spill out of the tank? (b) Find the total force at back and front ends of the tank after the spilling of the water out of tank.

4.6 A closed rectangular tank 10 m long, 5 m wide and 3 m deep is completely filled with an oil of specific gravity 0.92. Find the pressure difference between the rear and front top corners of the tank, if it is moving with an acceleration of 3 m/s^2 in the horizontal direction.

4.7(a) An open rectangular tank 4 m long and 2.5 m wide contains an oil of specific gravity 0.85 up to a depth of 1.5 m. Determine the total pressure on the bottom of the tank, when the tank is moving with an acceleration of g/2 m/s^2, (i) vertically upward, and (ii) vertically downward.

(b) Same as Problem 4.7(a), but now the tank is moving with an acceleration of g m/s^2, (i) vertically upward, and (ii) vertically downward.

4.8 A cylindrical tank 2 m high and 1 m in diameter contains 1.5 m of water. What are the pressures at the bottom of the tank at the center and at the walls when the tank rotates at a constant angular velocity of 6 rad/s ?

4.9 A cylinder 15 cm diameter and 37.5 cm long containing water is rotated about its vertical axis at a speed of 320 rpm so that a portion of water spills out. If the cylinder is brought to rest, what would be the depth of water in it ?

****

Chapter 5

FLUID KINEMATICS

5.1 INTRODUCTION

Kinematics is the geometry of motion. Thus, the kinematics of fluids describes the fluid motion (velocity and acceleration of fluid particles) without considering the forces which caused that motion. Kinematics is important because it can explain many fluid phenomena in a simple way.

5.2 DESCRIPTION OF FLUID MOTION

A fluid consists of an innumerable number of particles, each of which has its own velocity and acceleration, which may change both with respect to time and space. So, for a complete analysis of fluid motion, it is necessary to observe the motion of the fluid particles at various points and times. Two methods are generally used for the mathematical analysis of the fluid motion:

1. **Lagrangian method:** It deals with the study of the flow pattern of the individual particles. In this method, the path traced by the particle under consideration is studied in detail.

2. **Eulerian method:** It deals with the study of the flow pattern of all the particles simultaneously as they pass fixed points in space. In this method, the paths traced by all the particles at one section and one time are studied in detail.

The general example, to explain both the methods, is the study of movement of a number of vehicles on a busy road. The Lagrangian method deals with the study of the movement of only one vehicle through a specified distance. The Eulerian method deals with the study of movement of all the vehicles on the road at one section and at one instant.

The Eulerian method is commonly used because of its mathematical simplicity. Moreover, in Fluid Mechanics the movement of an individual fluid particle is not of much importance.

5.3 TYPES OF FLOW LINES

Whenever a fluid is in motion, its particles move along certain lines depending upon the flow conditions. The following lines are considered suitable to describe the fluid motion.

Path Line: The path traced by a single fluid particle in motion is called a path line. Thus, the path line shows the direction of the velocity of a particle at successive instant if time.

Stream Line: The imaginary line drawn in the fluid such that the tangent at any point on the line indicates the direction of velocity of the fluid particles at that point is known as a stream line (Fig.5.1). Since at any point the velocity is tangential to the stream line, so the component of velocity at right angles to the stream line is always zero. Thus, there can be no flow occurring across a stream line. No two streamlines can ever cross one another.

Streamtube: An element of fluid bounded by a number of stream lines which confine the flow is called a streamtube (Fig. 5.2). Since the velocity normal to a stream line is zero, so no flow can enter or leave the streamtube except at the ends. Therefore, streamtube behaves like a solid tube.

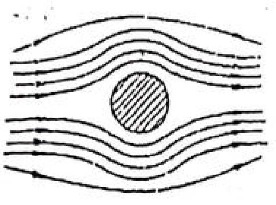


Fig. 5.1 Stream lines

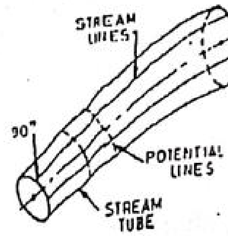


Fig. 5.2 Streamtube

5.4 GENERAL TYPES OF FLUID FLOW

Flows are classified on the basis of different criteria as follows:

Steady and Unsteady Flows

Flow is said to be steady if the flow parameters like velocity, density, viscosity, pressure, surface tension, temperature, etc. at a particular position in space do not change with time. Mathematically, if P denotes a flow parameter, then for steady flow

$$\frac{\partial P}{\partial t} = 0 \qquad \text{for fixed position in space} \qquad (5.1)$$

In steady flow, the path lines and the stream lines coincide.

Flow is said to be unsteady if the flow parameters at any particular position in space change with time. Mathematically, for unsteady flow

$$\frac{\partial P}{\partial t} \neq 0 \qquad \text{for fixed position in space} \qquad (5.2)$$

Uniform and Non-uniform Flows

A uniform flow is one in which the flow parameters at any given instant remain same at every point in space. Mathematically, for uniform flow

$$\frac{\partial P}{\partial s} = 0 \qquad \text{for fixed time} \qquad (5.3)$$

where s denotes the direction in which flow occurs.

A flow is said to be non-uniform if the flow parameters at any instant change with distance. Mathematically, for non-uniform flow

$$\frac{\partial P}{\partial s} \neq 0 \qquad \text{for fixed time} \qquad (5.4)$$

Compressible and Incompressible Flows

A flow in which the volume and thereby the density of the fluid change appreciably during the flow is called a compressible flow. Gases are mostly compressible.

A flow in which the volume and thereby the density do not change appreciably during the flow is called an incompressible flow. Liquids are generally considered to be incompressible.

Rotational and Irrotational Flows

A flow in which the fluid particles rotate about their own mass centers while flowing, i.e. have some angular velocity normal to the plane of motion, is called a rotational flow. A flow in which the fluid particles do not rotate about their own mass centers while flowing and retain their original orientations is called an irrotational flow.

One, Two and Three-Dimensional Flows

In a Cartesian (x, y, z) coordinate system, the flow parameters like velocity, pressure, etc. may have components along all the three coordinate axes. One-dimensional flow is the one in which the flow parameters are functions of one coordinate only. In two-dimensional flow the flow parameters are functions of two coordinates and in three-dimensional flow the flow parameters are functions of three coordinate directions. It is to be noted that one, two and three-dimensional flows can be either steady or unsteady, depending upon whether the flow parameters change with time or not. Thus, if V is the velocity of a fluid particle, it can be expressed mathematically as follows.

Steady one-dimensional flow: $V = f(x)$
Steady two-dimensional flow: $V = f(x,y)$
Steady three-dimensional flow: $V = f(x,y,z)$
Unsteady one-dimensional flow: $V = f(x,t)$
Unsteady two-dimensional flow: $V = f(x,y,t)$
Unsteady one-dimensional flow: $V = f(x,y,z,t)$

5.5 DIFFERENT TYPES OF DISPLACEMENTS OF FLUID ELEMENTS

Any fluid element can be translated, rotated or distorted during its course of motion. Correspondingly, the displacements of fluid elements are called (i) translation, (ii) rotation, and (iii) distortion or deformation.

Translation: The fluid elements move bodily in some direction. For example, in flow through pipes of constant diameter, any element of fluid is simply moved from its original position to another position after some time (Fig. 5.3). A pure translation does not cause any stress in the element.

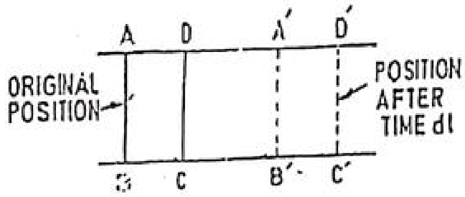


Fig. 5.3 Pure translation of fluid element

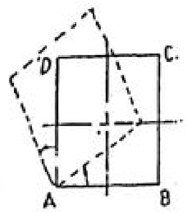


Fig. 5.4 Pure rotation of fluid element

Rotation: Pure rotation is shown in Fig. 5.4. Here, rotation of AB and AD are in the same direction, i.e. anticlockwise. In pure rotation, no stress is caused in the fluid element.

Distortion or Deformation: The distortion is of two types: (i) angular distortion (Fig. 5.5), and (ii) volume or linear distortion (Fig. 5.6).

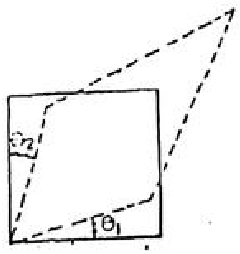


Fig. 5.5 Angular distortion of fluid element

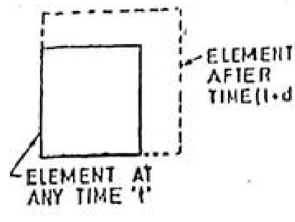


Fig. 5.6 Volume distortion of fluid element

The angular distortion consists of rotations both in the anticlockwise and clockwise directions (θ_1 is anticlockwise and θ_2 is clockwise in Fig. 5.5), volume remaining the same. In volume distortion, volume is changed (Fig. 5.6). In case of distortion of a fluid element, the stresses are produced in it.

5.6 ROTATION AND VORTICITY

A flow is said to be rotational if the fluid particles rotate about their own mass centers. If u, v and w are the components of the velocity of a fluid particle at any point in the flow in the x, y and z directions, respectively, then the rotation about x, y and z axes are given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad (5.5a)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad (5.5b)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (5.5c)$$

where the subscript x for ω is used for the rotation about the x-axis, the motion being in the y-z plane.

The resultant rotation vector $\overline{\omega}$ is given by

$$\overline{\omega} = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} \quad (5.6)$$

Vorticity is equal to twice the rotation. Therefore, the vorticity vector designated by $\overline{\Omega}$ is given by

$$\overline{\Omega} = 2\overline{\omega} \quad (5.7)$$

The three components of vorticity are therefore

$$\Omega_x = 2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad (5.8a)$$

$$\Omega_y = 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad (5.8b)$$

$$\Omega_z = 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (5.8c)$$

For irrotational flow, the net rotation is zero, i.e. $\omega = 0$. This implies that

$$\omega_x = \omega_y = \omega_z = 0 \quad (5.9)$$

or

$$\Omega_x = \Omega_y = \Omega_z = 0 \quad (5.10)$$

which gives the condition for irrotationality as

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad (5.11)$$

The rotation of fluid particles will always be associated with shear stress. Hence, fluids of more viscosity where shear stress plays an important part are all rotational and fluids of less viscosity, e.g. water and air, are of irrotational type.

The circulation refers to flow around a closed path. It is denoted by Γ. The relation between vorticity and circulation is given by

$$\Omega = \frac{\Gamma}{A} \quad (512)$$

, or

$$\Gamma = \Omega A \quad (5.13)$$

i.e. vorticity is the circulation per unit area. The units of circulation is then rad/s$\times$m^2 = m^2/s.

Example 5.1

The velocity components at a point in a flow in the x, y and z directions are respectively u = a + by – cz, v = d – bx – ez and w = f + cx – ey, where a, b, c, d, e and f are arbitrary constants. Does it represent rotational flow? If so, determine rotation and vorticity.

Solution The components of rotation are

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}[-c-(-c)] = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}[-c-c] = -c$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[-b-b] = -b$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{0+(-c)^2+(-b)^2} = \sqrt{b^2+c^2} \quad \text{rad/s}$$

Since $\omega \neq 0$, the flow is rotational and $\omega = \sqrt{b^2+c^2}$ rad/s. Now, since vorticity = 2 × rotation, so

$$\Omega_x = 2\omega_x = 0,\ \Omega_y = 2\omega_y = -2c,\ \Omega_z = 2\omega_z = -2b$$

$$\therefore \Omega = \sqrt{\Omega_x^2 + \Omega_y^2 + \Omega_z^2} = \sqrt{0^2 + (-2c)^2 + (-2b)^2} = 2\sqrt{b^2+c^2} \quad \text{rad/s}$$

5.7 EQUATION OF STREAM LINE

In order to know the type of fluid motion, it is required to derive the equation of stream line. Consider a stream line OA in a two-dimensional flow and let the velocity of a fluid particle at point O be $\overline{V}$ having its components u and v in the x and y directions, respectively (Fig. 5.7). After a time dt, the distance moved by the particle in the x and y directions will be

udt = dx and vdt = dy

Then,

dx : dy = udt : vdt = u : v

or

$$\frac{dy}{dx} = \frac{v}{u} \qquad (5.14)$$

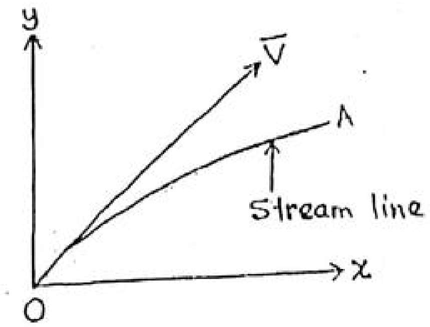


Fig. 5.7 Equation of stream line

This equation gives the tangent to the stream line at O. It is known as the equation of the stream line. It can also be written as

$$u\,dy - v\,dx = 0 \qquad (5.15)$$

5.8 STREAM FUNCTION

In a two-dimensional flow, the stream function ψ is a function of x and y such that

$$u = \frac{\delta\psi}{\delta y} \qquad (5.16)$$

$$v = -\frac{\delta\psi}{\delta x} \qquad (5.17)$$

where u and v are the components of velocity $\overline{V}$ in the x and y directions, respectively.

Putting the values of u and v in Eq. (5.15), we obtain

$$\frac{\delta\psi}{\delta y}dy + \frac{\delta\psi}{\delta x}dx = 0 \qquad (5.18)$$

Since ψ is the function of x and y, the left hand side of this equation is equal to the total differential dψ. Therefore,

$$d\psi = 0 \tag{5.19}$$

or

$$\psi = \text{constant for a stream line} \tag{5.20}$$

The discharge per unit width between two stream lines will be the difference in the values of the two stream functions, i.e.

$$q = \psi_2 - \psi_1 \tag{5.21}$$

In a two-dimensional flow, the resultant velocity V at any point P(x,y) is given by the relation

$$V = \sqrt{u^2 + v^2} \tag{5.22}$$

Example 5.2

For a two-dimensional flow, the stream function is given by $\psi = 2xy$. Calculate the velocity at the point (3, 6).

Solution If u and v are the velocity components in the x and y directions, respectively, then

$$u = \frac{\delta\psi}{\delta y} = \frac{\delta}{\delta y}(2xy) = 2x = 2 \times 3 = 6 \text{ m/s}$$

$$v = -\frac{\delta\psi}{\delta x} = -\frac{\delta}{\delta y}(2xy) = -2y = -2 \times 6 = -12 \text{ m/s}$$

$\therefore$ The resultant velocity at the point (3,6) is obtained as

$$V = \sqrt{u^2 + v^2} = \sqrt{6^2 + (-12)^2} = 13.42 \text{ m/s}$$

5.9 VELOCITY POTENTIAL

Velocity potential ϕ is a function of x, y and z such that its partial derivative with respect to displacement in any direction is equal to the velocity component in that direction. Mathematically,

$$u = \frac{\partial\phi}{\partial x},\ v = \frac{\partial\phi}{\partial y},\ w = \frac{\partial\phi}{\partial z} \tag{5.23}$$

where u, v and w are the velocity components in the x, y and z directions, respectively.

The relationship between the stream function and the velocity potential for two-dimensional steady, irrotational and incompressible flow is obtained from Eqs.(5.16), (5.17) and (5.23) as follows:

$$u = \frac{\partial\phi}{\partial x} = \frac{\delta\psi}{\delta y} \tag{5.24}$$

$$v = \frac{\partial\phi}{\partial y} = -\frac{\delta\psi}{\delta x} \tag{5.25}$$

Potential line is a line along which the velocity potential ϕ is constant. The potential lines are orthogonal to stream lines as is shown below.

Since $\phi = f(x,y)$ for steady flow in two dimensions, so

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

But along a potential line, $d\phi = 0$.

$$\therefore \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$$

or

$$\frac{dy}{dx} = -\frac{\partial\phi}{\partial x}\bigg/\frac{\partial\phi}{\partial y} = -\frac{u}{v}$$

∴ Slope of the potential line = - u/v.

Again, $\psi = f(x, y)$ for steady flow in two dimensions. So

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

But along a stream line, $d\psi = 0$.

$$\therefore \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$ => $\frac{\partial \psi}{\partial y} dy = -\frac{\partial \psi}{\partial x} dx$

or,

$$\frac{dy}{dx} = -\frac{\partial \psi}{\partial y} / \frac{\partial \psi}{\partial x} = \frac{v}{u}$$ => $\frac{dy}{dx} = -\frac{\partial \psi}{\partial x} / \frac{\partial \psi}{\partial y} = v/u$

∴ Slope of the stream line = v/u.

Thus, slope of the potential line × slope of the stream line = $-\frac{u}{v} \times \frac{v}{u} = -1$, which shows that the potential lines and the stream lines are orthogonal, i.e. perpendicular to one another. -> Proof that (Q.)

Example 5.3

A stream function is given by the expression $\psi = 3x^2 - 3y^2$. Find the velocity potential.

Solution

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(3x^2 - 3y^2) = -6x$$

Applying Eq.(5.23), we obtain

$$\frac{\partial \phi}{\partial x} = u = -6y$$ -> integration

$$\therefore \phi = -6xy + f_1(y) \qquad \text{(i)}$$

and

$$\frac{\partial \phi}{\partial y} = v = -6x$$

$$\therefore \phi = -6xy + f_2(x) \qquad \text{(ii)}$$

From (i) and (ii), we get

$$\therefore \phi = -6xy + f_1(y) + f_2(x)$$

5.10 ACCELERATION OF FLUID PARTICLES

The acceleration of a fluid particle is the time rate of change of its velocity vector. Let a be the acceleration at point P(x,y,z) and a_x, a_y and a_z denote the components of the acceleration in x, y and z directions, respectively. Then,

$$a_x = \frac{du}{dt}, \qquad a_y = \frac{dv}{dt}, \qquad a_z = \frac{dw}{dt} \qquad (5.26)$$

and

$$a=\sqrt{a_x^2+a_y^2+a_z^2} \tag{5.27}$$

If $\bar{V}(x,y,z,t)$ is the velocity vector and $\bar{a}$ is the acceleration vector, then

$$\bar{a}=\frac{d\bar{V}}{dt}(x,y,z,t) \tag{5.28}$$

Now,

$$d\bar{V}=\frac{\partial \bar{V}}{\partial t}dt+\frac{\partial \bar{V}}{\partial x}dx+\frac{\partial \bar{V}}{\partial y}dy+\frac{\partial \bar{V}}{\partial z}dz \tag{5.29}$$

$$\therefore \frac{d\bar{V}}{dt}=\frac{\partial \bar{V}}{\partial t}+\frac{\partial \bar{V}}{\partial x}\frac{dx}{dt}+\frac{\partial \bar{V}}{\partial y}\frac{dy}{dt}+\frac{\partial \bar{V}}{\partial z}\frac{dz}{dt} \tag{5.30}$$

$$\therefore \bar{a}=\frac{\partial \bar{V}}{\partial t}+\frac{\partial \bar{V}}{\partial x}\frac{dx}{dt}+\frac{\partial \bar{V}}{\partial y}\frac{dy}{dt}+\frac{\partial \bar{V}}{\partial z}\frac{dz}{dt}=\frac{\partial \bar{V}}{\partial t}+u\frac{\partial \bar{V}}{\partial x}+v\frac{\partial \bar{V}}{\partial y}+w\frac{\partial \bar{V}}{\partial z} \tag{5.31}$$

In Cartesian coordinates

$$a_x=\frac{\partial u}{\partial t}+(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}) \tag{5.32a}$$

$$a_y=\frac{\partial v}{\partial t}+(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}) \tag{5.32b}$$

$$a_z=\frac{\partial w}{\partial t}+(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}) \tag{5.32c}$$

The above set of equations indicates that the acceleration can result (i) from change with time at a given point in space, and (ii) from change in position from one point to another. The acceleration which is due to change with time at a point is known as the *local acceleration* and is given by the terms $\partial u/\partial t$, $\partial v/\partial t$ and $\partial w/\partial t$. The acceleration which is due to change in position is known as the *convective acceleration* and is represented by the three terms within the parentheses.

If the local acceleration is zero, the flow is called steady flow. If the convective acceleration is zero, the flow is called uniform flow. So, in steady uniform flow there is no acceleration.

5.11 FLOW NET

It is a graphical representation of stream lines and potential lines, which are orthogonal. The stream lines show the direction of flow. The potential lines show the lines joining the points of equal velocity potential. The flow net can be drawn only for irrotational flow. It helps in depicting and analyzing the behavior of irrotational flow. Certain flow phenomena which cannot be easily analyzed by mathematical means, may be analyzed and studied by drawing flow nets, e.g. flow through a Francis water turbine runner.

Consider a flow net where the stream lines are diverging as shown in Fig. 5.8. Consider two sections of the flow net 1 and 2. Let d_1 be the spacing between two stream lines at section 1, d_2 be the spacing between two stream lines at section 2, v_1 be the velocity of fluid particles at section 1 and v_2 be the velocity of fluid particles at section 2. Since there can be no flow across the stream lines, therefore the discharge per unit width between two consecutive stream lines will be equal, i.e.

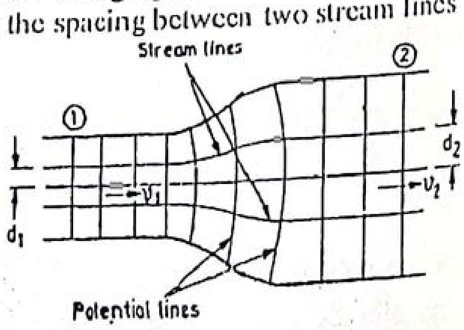


Fig. 5.8 Flow net

$$q=d_1v_1=d_2v_2 \tag{5.33}$$

It is thus obvious that the velocity of fluid particles varies inversely with the spacing between the stream lines, i.e. velocity decreases with increase in stream line spacing and vice versa.

Example 5.4

From a flow net diagram, it was found that the distance between two consecutive stream lines at two successive sections are 10 mm and 6 mm, respectively. If the velocity at the first section is 1 m/s, find the velocity at the other section. Also, find the discharge between the two stream lines.

Solution We have, $d_1 = 10$ mm $= 1$ cm, $d_2 = 6$ mm $= 0.6$ cm and $v_1 = 1$ m/s $= 100$ cm/s

$\therefore$ Using the relation, $d_1 v_1 = d_2 v_2$, we have

$$1 \times 100 = 0.6 \times v_2$$

or

$$v_2 = \frac{100}{0.6} = 167 \text{ cm/s}$$

The discharge between the streamlines is given by

$$q = d_1 v_1 = 1 \times 100 = 100 \text{ cm}^2/\text{s}$$

PROBLEMS AND EXERCISES

5.1 Describe the two methods which are generally used for the mathematical analysis of fluid motion. Which one is most commonly used and why?

5.2 Write short notes on (i) path line, (ii) stream line, (iii) streamtube, (iv) steady flow, (v) unsteady flow, (vi) uniform flow, (vii) non-uniform flow, (viii) compressible flow, (ix) rotational flow, (x) rotation and vorticity, (xi) stream function, (xii) velocity potential, (xiii) potential line, (xiv) local and convective accelerations, and (xv) flow net.

5.3 Show that the stream lines and the potential lines are orthogonal.

5.4 For a two-dimensional flow, the stream function is given by $\psi = 2x^2 - y^2$. Calculate the velocity at the point (2,3).

5.5 A stream function is given by the expression $\psi = 2xy$. Prove that the flow is irrotational.

5.6 The velocity along a stream line is given by $v = 2s + t + 3$. What would be the convective and local accelerations after 1 sec when $s = 2$?

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Chapter 6

FLUID DYNAMICS

6.1 INTRODUCTION

Fluid kinematics deals with the motion of the fluid particles without considering the forces which caused the motion. Fluid dynamics deals with the motion of fluids and the forces causing the motion.

To describe the fluid motion, a set of equations should be available which can be solved analytically or numerically applying appropriate initial and boundary conditions. The three basic equations which describe fluid motion are

i) the equation of continuity based on the principle of conservation of mass,
ii) the equation of energy based on the principle of conservation of energy, and
iii) the equation of momentum based on the principle of conservation of momentum.

6.2 EQUATION OF CONTINUITY

Discharge and Mean Velocity

The volume of fluid passing per unit time across a section of a conduit (a pipe or a channel) is known as discharge or volume flow rate. It is generally denoted by Q and expressed in m^3/s or cumec. Let A be the cross-sectional area of the conduit and V is the cross-sectional mean velocity of flow. Then, the discharge is given by

$$Q = \text{Cross-sectional area} \times \text{Cross-sectional mean velocity} = AV \qquad (6.1)$$

where A is in m^2, V is in m/s and Q is in m^3/s.

For pipe and channel flows, it is more convenient to use the cross-sectional mean or average velocity than the point velocity. When the velocity varies over the cross-section of a conduit, the mean velocity is obtained as follows. Let v be the velocity of a liquid over the elementary area dA and V be the mean or average velocity over the entire cross-section having an area A. Then

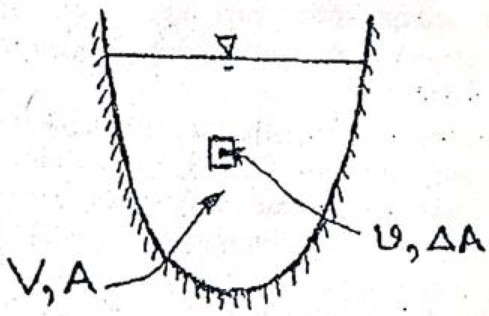


Fig. 6.1 Determination of discharge and mean velocity

$$Q = AV = \int_0^A v\,dA \qquad (6.2)$$

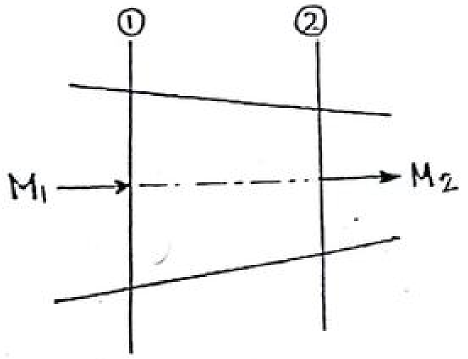
or

$$V = \frac{Q}{A} = \frac{1}{A}\int_0^A v\,dA \qquad (6.3)$$

Equation of Continuity for One-Dimensional Steady Flow

The equation of continuity is derived from the principle of conservation of mass which states that in the non-nuclear process matter can neither be created nor destroyed. So, the mass of fluid per unit time entering the upstream section of a conduit must be equal to the mass of fluid per unit time at the downstream section plus the mass of fluid stored or accumulated in between the two sections. In a steady flow there cannot be any storage or accumulation of mass in between the two sections. So, the mass flow rates at the two sections must be the same.

Consider the flow of a fluid (e.g. water) between sections 1 and 2 of a pipe (Fig. 6.2) Let A_1 be the cross-sectional area, V_1 be the mean velocity, M_1 be the mass of fluid entering per unit time, ρ_1 is the mass density and Q_1 is the discharge at section 1. The corresponding quantities at section 2 are A_2, V_2, M_2, ρ_2 and Q_2, respectively. Then, when the flow is steady, according to the principle of conservation of mass, we can write



$$M_1 = M_2 \qquad (6.4)$$

Fig. 6.2 One-dimensional steady flow through a pipe

Since mass = mass density × volume, so Eq.(6.4) can be written as

$$\rho_1 Q_1 = \rho_2 Q_2 \qquad (6.5)$$

which is the general equation of continuity for one-dimensional steady flow. It is the first and fundamental equation of flow.

For an incompressible fluid (e.g. water), $\rho_1 = \rho_2$. Hence, Eq.(6.5) reduces to

$$Q_1 = Q_2 \qquad (6.6)$$

or

$$A_1 V_1 = A_2 V_2 \qquad (6.7)$$

Equation (6.6) or (6.7) implies that for steady one-dimensional incompressible flow, the discharge remains the same at all sections of a conduit, provided fluid is neither injected or taken out of it.

The equation of continuity, Eq.(6.6) or (6.7), is applicable only when

(1) the flow is steady which is usually the case for most of the problems of fluid mechanics,

(2) the density is constant, i.e. the flow is incompressible, which is the case for most problems of hydraulics where compressibility effect is negligible, and

(3) the flow is one-dimensional. All pipe and channel flow problems are solved by this assumption, because of simplicity and practical purposes.

Example 6.1

Water is flowing through a pipe 10 cm in diameter with an average velocity of 10 m/s. Compute the discharge in liters/sec. Also, determine the velocity at the other end of the pipe, if the diameter of the pipe is gradually changed to 20 cm.

Solution Using the relation Q = AU, we obtain

$$Q = \frac{\pi}{4}\left(\frac{10}{100}\right)^2 \times 10 = 0.07854\ m^3/s = 0.07854 \times 10^3 \text{ liters/sec} = 78.54 \text{ liters/sec}$$

Again, using the relation $A_1V_1 = A_2V_2$, we obtain the velocity at the other end of the pipe as

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{\frac{\pi}{4}\left(\frac{10}{100}\right)^2 \times 10}{\frac{\pi}{4}\left(\frac{20}{100}\right)^2} = \frac{10}{4} = 2.5\ m/s$$

Equation of Continuity for One-Dimensional Unsteady Flow

Consider the flow of a fluid in a small length Δs of a conduit (Fig. 6.3). Let ρ, A and V be the mass density, cross-sectional area and the mean velocity at section 1. Now, mass of fluid entering the control volume in time Δt

$$= \rho A V \Delta t$$

and mass of fluid leaving the control volume in time Δt

$$= \left[\rho A V + \frac{\partial}{\partial s}(\rho A V)\Delta s\right]\Delta t$$

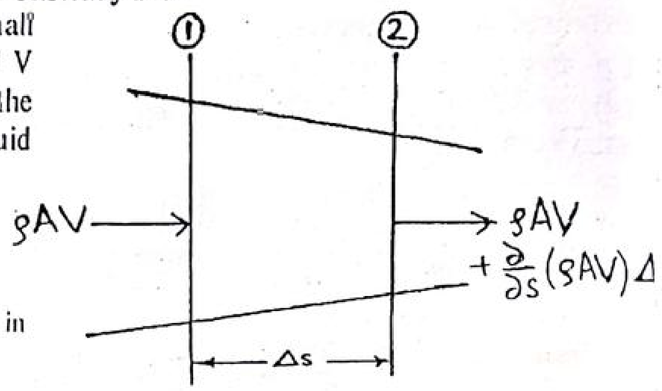


Fig. 6.3 One-dimensional unsteady flow in a conduit

$\therefore$ Increase of mass in the control volume in time Δt

$$= \rho A V \Delta t - \left[\rho A V + \frac{\partial}{\partial s}(\rho A V)\Delta s\right]\Delta t = -\frac{\partial}{\partial s}(\rho A V)\Delta s \Delta t \quad \text{(i)}$$

But increase in mass of the control volume due to storage in time Δt

$$= \frac{\partial}{\partial t}(\rho A \Delta s)\Delta t = \frac{\partial}{\partial t}(\rho A)\Delta s \Delta t \quad \text{(ii)}$$

$\therefore$ Equating (i) and (ii), we obtain

$$\frac{\partial}{\partial t}(\rho A)\Delta s \Delta t = -\frac{\partial}{\partial s}(\rho A V)\Delta s \Delta t$$

or

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho A V) = 0 \quad (6.8)$$

which is the equation of continuity for one-dimensional unsteady flow in differential form.

When the flow is steady, $\partial(\rho A)/\partial t = 0$ and, hence

$$\frac{\partial}{\partial s}(\rho A V) = 0$$

or $\quad \rho A V = \rho Q = \text{constant}$

or $\quad \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad$ or $\quad \rho_1 Q_1 = \rho_2 Q_2$

and when the flow is incompressible, then $\rho_1 = \rho_2$ and so

$Q_1 = Q_2 \quad$ or $\quad A_1 V_1 = A_2 V_2$

which is the same as derived earlier.

Equation of Continuity for Three-Dimensional Unsteady Flow

Consider a parallelepiped fluid element (or control volume) in the flow, with sides Δx, Δy and Δz in the x, y and z directions, respectively (Fig. 6.4). Let u, v and w be the components of velocity of the fluid at the centroid P(x, y, z) of the control volume in the three directions, respectively. Let ρ be the mass density of the fluid at time t.

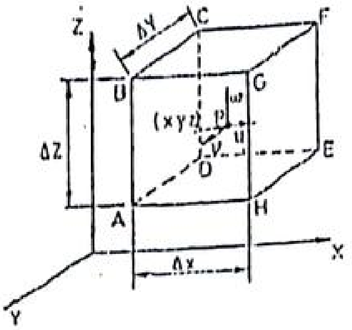


Fig. 6.4 Parallelepiped fluid element

Now, the mass of fluid entering across the face ABCD in time Δt is

$$\left[(\rho u\Delta y\Delta z)-\frac{\partial}{\partial x}(\rho u\Delta y\Delta z)\frac{\Delta x}{2}\right]\Delta t$$

and the mass of fluid leaving across the face EFGH in time Δt is

$$\left[(\rho u\Delta y\Delta z)+\frac{\partial}{\partial x}(\rho u\Delta y\Delta z)\frac{\Delta x}{2}\right]\Delta t$$

$\therefore$ Gain of fluid mass of the control volume in time Δt due to flow in the x-direction is

$$=\left[(\rho u\Delta y\Delta z)-\frac{\partial}{\partial x}(\rho u\Delta y\Delta z)\frac{\Delta x}{2}\right]\Delta t-\left[(\rho u\Delta y\Delta z)+\frac{\partial}{\partial x}(\rho u\Delta y\Delta z)\frac{\Delta x}{2}\right]\Delta t$$

$$=-\frac{\partial}{\partial x}(\rho u)\Delta x\Delta y\Delta z\Delta t$$

Similarly, gain of fluid mass of the control volume in time Δt due to flow in the y-direction is given by

$$=-\frac{\partial}{\partial y}(\rho v)\Delta x\Delta y\Delta z\Delta t$$

and gain of fluid mass of the control volume in time Δt due to flow in the z-direction is given by

$$=-\frac{\partial}{\partial z}(\rho w)\Delta x\Delta y\Delta z\Delta t$$

$\therefore$ Total gain of fluid mass of the control volume in time Δt becomes

$$=-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\Delta x\Delta y\Delta z\Delta t \qquad \text{(i)}$$

The increase in the mass of the control volume in time Δt is equal to

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)\Delta t=\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z\Delta t \qquad \text{(ii)}$$

$\therefore$ Equating (i) and (ii), we obtain

$$\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z\Delta t=-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\Delta x\Delta y\Delta z\Delta t$$

or

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (6.9)$$

This is the equation of continuity for three-dimensional unsteady flow.

Case I: Steady flow

For steady flow the density does not change with time, i.e. $\partial\rho/\partial t=0$. Therefore, Eq.(6.9) becomes

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \quad (6.10)$$

Case II: Steady incompressible flow

For steady incompressible flow the density does not change with x, y, z and t, i.e. $\partial\rho/\partial t=\partial\rho/\partial x=\partial\rho/\partial y=\partial\rho/\partial z=0$. Therefore, Eq.(6.9) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (6.11)$$

Case III: Two-dimensional flow

If the flow is in the x-y plane, then the last term of Eq. (6.11) will not exist, i.e. velocity component w = 0. Then, the equation of continuity for steady incompressible flow becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (6.12)$$

Example 6.2

The velocity distribution for a two-dimensional incompressible flow is given by u = ln x^2y^2 – 4 ln xt and v = 2y/x + 4 ln xt. Show that it satisfies the equation of continuity.

Solution The equation of continuity for two-dimensional flow is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Now, $\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(\ln x^2y^2-4\ln xt)=\frac{1}{x^2y^2}\times 2xy^2-4\times\frac{1}{xt}\times t=\frac{2}{x}-\frac{4}{x}=-\frac{2}{x}$

$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}(\frac{2y}{x}+4\ln xt)=\frac{2}{x}+0=\frac{2}{x}$

$\therefore \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{2}{x}+\frac{2}{x}=0$

Hence, the equation of continuity is satisfied.

6.3 ENERGY EQUATION

Energy in a Fluid Flow

The energy may be defined as the capacity to do work. It manifests in various forms and it can change from one form to another. The three forms of energy present in fluid flow are (i) potential energy, (ii) pressure energy, and (iii) kinetic energy.

Potential energy: This energy in a fluid exists by virtue of its position or elevation with respect to a horizontal datum. If a body of fluid weighing W kg is z meters above the datum, its potential energy will be W.z m-kg or m-N. Thus, a body of fluid having unit weight (i.e. W = 1 kg) and z meters above datum has potential energy of z m-kg/kg or m-N/N or simply z m. Therefore, z is called the potential or elevation or datum head, i.e. the potential or elevation or datum head is z meters of fluid.

Pressure energy: It is the energy possessed by a fluid by virtue of its existing pressure. If a particle of a fluid is under a pressure of p kg/m², then the pressure energy of the particle will be p/γ m-kg/kg or m-N/N or simply m, when γ is the specific weight of the fluid. Therefore, p/γ is called the pressure head, i.e. the pressure head is p/γ m of fluid.

Kinetic energy: It is the energy possessed by a fluid particle by virtue of its velocity. If every particle of a fluid mass M moves with uniform velocity V, then the kinetic energy will be ½MV^2. Since the weight W = Mg, the kinetic energy in terms of W will be $WV^2/2g$. For a unit weight, the kinetic energy is $V^2/2g$ m-kg/kg or m-N/N or simply m. Therefore, $V^2/2g$ is known as velocity or kinetic energy head, i.e. the velocity or kinetic energy head is $V^2/2g$ m of fluid.

Total energy or head of a fluid particle in motion: The total energy of a fluid particle in motion is the sum of its potential energy, pressure energy and kinetic energy. Mathematically,

$$\text{Total energy, } H = z + \frac{p}{\gamma} + \frac{V^2}{2g} \quad \text{m-kg/kg or m-N/N} \tag{6.13a}$$

and the total head of a fluid particle in motion is the sum of its potential energy head, pressure head and kinetic energy head. Mathematically,

$$\text{Total head, } H = z + \frac{p}{\gamma} + \frac{V^2}{2g} \quad \text{m of fluid} \tag{6.13b}$$

Frictional loss of head: Every real fluid has some viscosity. It resists the relative motion of flowing fluid. During the flow of real fluids, the viscosity and turbulence consume a fraction of energy which takes the form of heat or thermal energy. This form of energy is not reversible and is not recoverable as useful energy, though it slightly raises the fluid temperature. It is a loss of useful energy due to viscosity and turbulence of real fluid. It is therefore called frictional loss of head and denoted by h_f.

Example 6.3

Water is flowing through a pipe 7 cm in diameter under a gauge pressure of 3.5 kg/cm² and with a mean velocity of 1.5 m/s. Neglecting friction, determine the total head, if the pipe is 7 m above the datum line.

Solution Given, pressure, $p = 3.5 \text{ kg/cm}^2 = 3.5\times10^4 \text{ kg/m}^2$

Velocity of water, V = 1.5 m/s

Datum head, z = 7 m

Specific weight, γ = 1000 kg/m³

$$\therefore \text{Total head, } H = z + \frac{p}{\gamma} + \frac{V^2}{2g} = 7 + \frac{3.5\times10^4}{1000} + \frac{1.5^2}{2\times9.81} = 42.12 \quad \text{m of water}$$

Bernoulli's Equation

Daniell Bernoulli (1700-1782), a Swiss mathematician, presented an equation in his book entitled *Hydrodunamica* in 1738. This equation is known as the Bernoulli's equation and can be stated as follows: *"In a steady flow of frictionless incompressible fluid, the total energy remains constant."* Mathematically,

$$z + \frac{p}{\gamma} + \frac{V^2}{2g} = \text{constant} \tag{6.14}$$

where, z = potential energy, p/γ is the pressure energy and $V^2/2g$ = kinetic energy.

A similar but more general statement of Bernoulli's equation derived from the general energy equation is: "*In a steady flow of frictionless incompressible fluid, the sum of elevation, pressure and velocity heads remains constant at every section, provided energy is neither added nor taken out by external source.*"

Another statement of Bernoulli's theorem as derived from the Euler equation is: "*In a steady flow of frictionless incompressible fluid, the sum of elevation, pressure and velocity heads remains constant along a stream line, provided energy is neither added nor taken out by external source.*"

Proof: Consider an incompressible liquid flowing through a pipe as shown in Fig. 6.5. Take two sections AA and BB of the pipe. Assume that the pipe is running full and the flow is continuous between the two sections. Suppose z_1 is the elevation of AA above the datum, p_1 is the pressure at AA, V_1 is the velocity of liquid at AA and A_1 is the area of the pipe at AA. Let z_2, p_2, V_2, and A_2 are the corresponding values at BB.

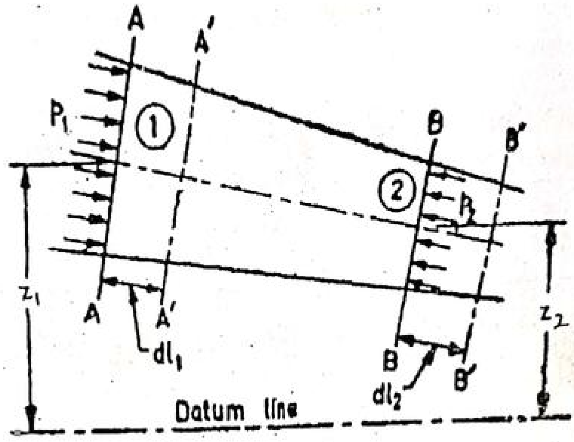


Fig. 6.5 Derivation of Bernoulli's equation

Let the liquid between the two sections AA and BB move to $A'A'$ and $B'B'$ through very small lengths dl_1 and dl_2. This movement of the liquid between AA and BB is equivalent to the movement of the liquid between AA and $A'A'$ to BB and $B'B'$, the remaining liquid between $A'A'$ and BB remaining unaffected.

Let W be the weight of liquid between AA and $A'A'$. Since the flow is continuous, so

$$W = \gamma A_1 dl_1 = \gamma A_2 dl_2$$

$$\therefore A_1 dl_1 = A_2 dl_2 = \frac{W}{\gamma} \tag{6.15}$$

Work done by pressure at AA in moving the liquid to $A'A'$

$$= \text{Force} \times \text{distance} = p_1 A_1 dl_1$$

and work done by pressure at BB in moving the liquid to $B'B'$

$$= -p_2 A_2 dl_2$$

Minus sign is taken as the direction of p_2 is opposite to that of p_1.

$\therefore$ Total work done by the pressure $= p_1 A_1 dl_1 - p_2 A_2 dl_2 = p_1 A_1 dl_1 - p_2 A_1 dl_1$

$$= A_1 dl_1 (p_1 - p_2) = \frac{W}{\gamma}(p_1 - p_2) \quad \text{(using Eq. (6.15))} \tag{6.16}$$

Loss of potential energy

$$= W(z_1 - z_2)$$

and gain in kinetic energy

$$W\left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right)$$

Since, loss in potential energy + work done by pressure = Gain in kinetic energy, so

$$W(z_1 - z_2) + \frac{W}{\gamma}(p_1 - p_2) = W\left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right)$$

or

$$z_1 - z_2 + \frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

or

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \tag{6.17}$$

or

$$z + \frac{p}{\gamma} + \frac{V^2}{2g} = \text{constant} \tag{6.18}$$

which proves the Bernoulli's equation.

Limitations of the Bernoulli's equation: The Bernoulli's equation has the following limitations:

1. Flow is steady.
2. Fluid is incompressible, i.e. the mass density is constant.
3. Fluid is non-viscous, i.e. frictional losses are zero.
4. Velocity is uniform over the section.
5. Except the gravity and pressure forces, no other forces are involved.

Euler's Equation of Motion

Leonhard Euler (1707–1788), a Swiss mathematician, gave an equation for steady flow of a non-viscous (frictionless) fluid along a stream line based on Newton's second law of motion. This is known as the Euler's equation of motion. The integration of this equation gives the Bernoulli's equation in the form of energy per unit weight of the flowing fluid. The Euler's equation is based on the following assumptions:

1. Flow is steady.
2. Fluid is incompressible, i.e. the mass density is constant.
3. Fluid is non-viscous, i.e. frictional losses are zero.
4. Velocity is uniform over the section.
5. Except the gravity and pressure forces, no other force or energy is involved.
6. Flow is along a stream line.

Let us consider a fluid element of cross-sectional area ΔA and length Δs on a stream line in a steady non-viscous (frictionless) flow (Fig. 6.6). Then there can be only two kinds of

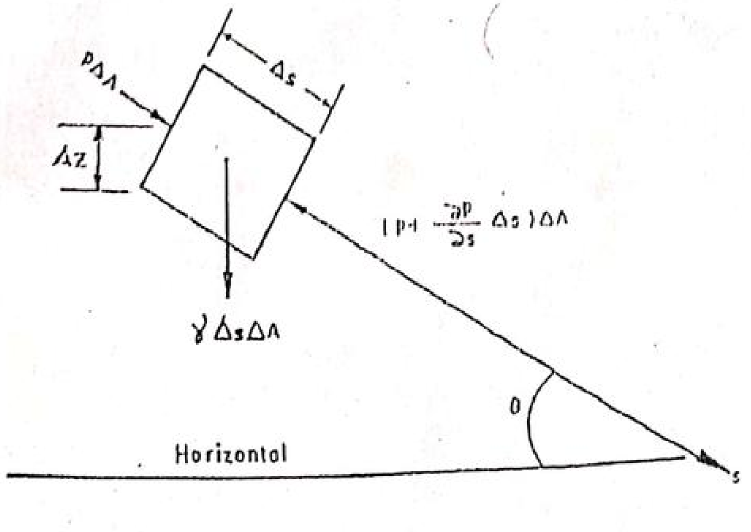


Fig. 6.6 Forces on a fluid element

force in any chosen direction s; that due to pressure gradient, i.e. $-(\partial p/\partial s)\Delta s\Delta A$ and that due to the weight of the element, i.e. $(\gamma\Delta s\Delta A\sin\theta)$ or $-(\gamma\Delta s\Delta A)\,\partial z/\partial s$, where z is the vertical height above datum and γ is the specific weight of the fluid. Let ρ be the mass density of the fluid and a_s be the acceleration in the s direction. Then, according to Newton's second law of motion, in any direction, force = mass × acceleration, we have

$$-\frac{\partial p}{\partial s}\Delta s\Delta A-\gamma\Delta s\Delta A\frac{\partial z}{\partial s}=(\rho\Delta s\Delta A)a_s$$

or

$$\frac{\partial}{\partial s}(p+\gamma z)+\rho a_s=0 \qquad (6.19)$$

Now, from the theory of partial differentiation, we can write

$$\frac{dV}{dt}=a_s=\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial s} \qquad (6.20)$$

When the flow is steady, $\partial V/\partial t=0$, and Eq. (6.19) becomes

$$\frac{\partial}{\partial s}(p+\gamma z)+\rho V\frac{\partial V}{\partial s}=0 \qquad (6.21)$$

which is the one-dimensional Euler equation of motion and was first developed by Euler in 1750.

Equation (6.21) can be integrated directly to give

$$p+\gamma z+\frac{1}{2}\rho V^2=\text{constant}$$

or

$$H=z+\frac{p}{\gamma}+\frac{V^2}{2g}=\text{constant} \qquad (6.22)$$

which is the *Bernoulli equation*. The Bernoulli expression $(z+p/\gamma+V^2/2g)$ in general varies from one streamline to another but remains constant along a streamline in steady non-viscous (frictionless) flow.

Kinetic Energy Coefficient

In deriving the Bernoulli's equation, the velocity has been assumed to be uniform over the entire cross-section and the mean or average velocity for the cross-section is used to compute the velocity or kinetic energy head. In the one-dimensional method of flow analysis, owing to non-uniform velocity distribution in a cross-section, the kinetic energy of flow computed from the cross-sectional mean velocity is generally less than its actual value. To get the actual kinetic energy of flow, the kinetic energy based on the mean velocity is multiplied by the coefficient α, known as the *kinetic energy coefficient* or the *kinetic energy correction factor* or the *Coriolis coefficient*.

Referring to Fig. 6.1, the kinetic energy of flow passing ΔA per unit time is equal to

$$\frac{1}{2}\times\rho v\Delta A\times v^2=\frac{\rho}{2}v^3\Delta A$$

where ρ is the mass density of water. Therefore, the total kinetic energy of flow passing the cross-section is equal to

$$\frac{\rho}{2}\int_0^A v^3 dA$$

where A is the total area of the cross-section. The total kinetic energy based on the mean velocity V and corrected for the non-uniform distribution of velocity is

$$\alpha\frac{\rho}{2}V^3A$$

Equating the above two quantities and reducing

$$\alpha = \frac{\int_0^A v^3 dA}{V^3 A} \qquad (6.23)$$

α is always positive

The kinetic energy coefficient is always positive and never less than unity. For uniform velocity distribution in the cross-section, α = 1. In all other cases, α > 1 and the further the velocity distribution departs from uniform, the greater the coefficient becomes. The effect of turbulence is to make the flow more uniform in the cross-section. Therefore, the value of α is higher for laminar flow than for turbulent flow.

For turbulent flow, the numerical value of α does not normally exceed 1.10 and one can assume α = 1 without any appreciable error. For laminar flow in a straight circular pipe, α = 2.

When the kinetic energy coefficient α is incorporated, the velocity or kinetic energy head becomes

$$\alpha \frac{V^2}{2g}$$

where V is the cross-sectional mean velocity, and the Bernoulli's equation takes the form

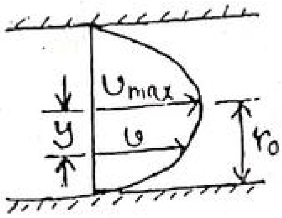
$$z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} \qquad (6.24)$$

Example 6.4

The velocity distribution in a turbulent flow in a pipe is given by Prandtl's one-seventh power law

$$\frac{v}{v_{max}} = \left(\frac{r_0 - y}{r_0}\right)^{1/7} \qquad (6.25)$$

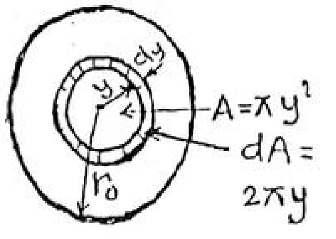
where y is the distance from the pipe wall and r_0 is the pipe radius. Determine the kinetic energy coefficient α.



Solution The mean velocity V is expressed by Eq. (6.3) as

$$V = \frac{1}{A}\int_0^A v dA = \frac{1}{\pi r_0^2}\int_0^{r_0} v.2\pi y dy = \frac{2\pi}{\pi r_0^2}\int_0^{r_0} yv dy = \frac{2}{r_0^2}\int_0^{r_0} y v_{max}\left(\frac{r_0 - y}{r_0}\right)^{1/7} dy$$

$$= \frac{2v_{max}}{r_0^2}\left(\frac{1}{r_0}\right)^{1/7}\int_0^{r_0}(r_0 y^{1/7} - y^{8/7})dy = \frac{2v_{max}}{r_0^2}\left(\frac{1}{r_0}\right)^{1/7}\left(r_0\frac{y^{8/7}}{8/7} - \frac{y^{15/7}}{15/7}\right)_0^{r_0}$$



$$= \frac{2v_{max}}{r_0^2}\left(\frac{1}{r_0}\right)^{1/7}\left(\frac{7r_0^{15/7}}{8} - \frac{7r_0^{15/7}}{15}\right) = \frac{2v_{max}}{r_0^2}\times r_0^2 \times \frac{49}{120} = \frac{98}{120}v_{max}$$

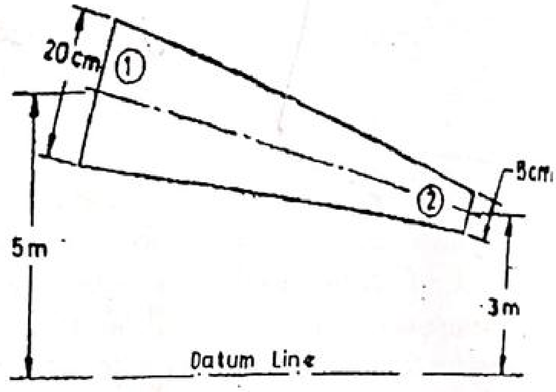
$$\therefore \alpha = \frac{1}{AV^3}\int_0^A v^3 dA = \frac{1}{\pi r_0^2}\times\left(\frac{120}{98}\right)^3\times\frac{1}{v_{max}^3}\times\int_0^{r_0} v_{max}^3\left(\frac{r_0 - y}{r_0}\right)^{3/7}\times 2\pi y dy$$

$$= \frac{2}{r_0^2}\times\left(\frac{120}{98}\right)^3\times\frac{1}{r_0^{3/7}}\times\int_0^{r_0}(r_0 y^{3/7} - y^{10/7})dy = \frac{2}{r_0^2}\times\left(\frac{120}{98}\right)^3\times\frac{1}{r_0^{3/7}}\left(\frac{7r_0^{17/7}}{10} - \frac{7r_0^{17/7}}{17}\right)$$

$$= 2\times\left(\frac{120}{98}\right)^3\times\frac{49}{170} = 1.058$$

Example 6.5

The diameter of a pipe changes from 20 cm at a section 5 m above datum to 5 cm at a section 3 m above datum. The pressure of water at the first section is 5 kg/cm^2. If the velocity of flow at the first section is 1 m/s, determine the intensity of pressure at the second section. Neglect losses.



Solution Area of the pipe at section 1

$$A_1 = \frac{\pi}{4} \times d_1^2 = \frac{\pi}{4} \times \left(\frac{20}{100}\right)^2 = \frac{\pi}{100} m^2$$

Area of the pipe at section 2

$$A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times \left(\frac{5}{100}\right)^2 = \frac{\pi}{1600} m^2$$

Velocity at section 1, $V_1 = 1$ m/s

Pressure at section 1, $p_1 = 5$ kg/cm$^2 = 5 \times 10^4$ kg/m^2

Elevation of pipe center at section 1, $z_1 = 5$ m

Elevation of pipe center at section 2, $z_2 = 3$ m

Specific weight of water, $\gamma = 1000$ kg/m^3

We know, $A_1 V_1 = A_2 V_2$

$$\therefore V_2 = \frac{A_1 V_1}{A_2} = \frac{\pi}{100} \times 1 \times \frac{1600}{\pi} = 16 \ m/s$$

Applying the Bernoulli's equation between sections 1 and 2

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

or

$$5 + \frac{5 \times 10^4}{1000} + \frac{1^2}{2 \times 9.81} = 3 + \frac{p_2}{1000} + \frac{16^2}{2 \times 9.81}$$

$$\therefore p_2 = (55.051 - 16.048) \times 1000 = 39003 \text{ kg/m}^2 = 3.90 \text{ kg/cm}^2$$

Energy Equation for One-Dimensional Steady Flow

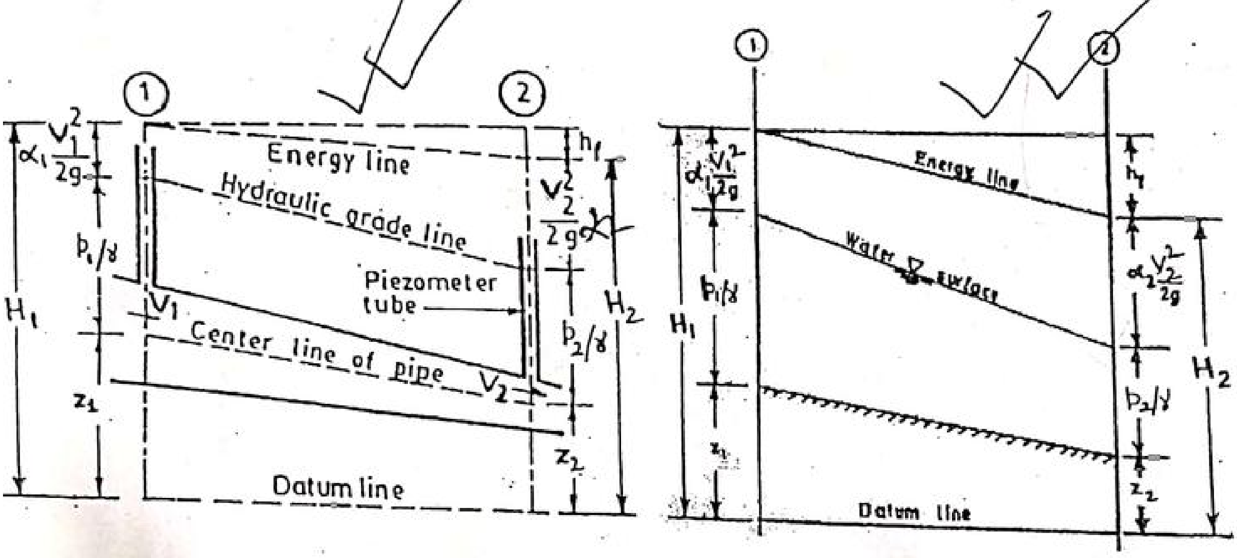
According to the principle of conservation of energy, the total energy at the upstream section 1 must be equal to the total energy at the downstream section 2 plus the frictional loss of energy h_f between the two sections (Fig. 6.7), i.e.

$$H_1 = H_2 + h_f$$

or

$$z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + h_f \qquad (6.26)$$

where α_1 and α_2 are the kinetic energy coefficients at sections 1 and 2, respectively. Equation (6.26) is known as the energy equation.



(a) Pipe flow

(b) Channel flow

Fig. 6.7 Definition sketch for the energy equation

Each term in the energy equation represents energy in m-kg/kg or m-N/N or simply m. The expression of energy in this form is particularly convenient for dealing with problems in pipes and channels.

The loss of energy may also be due to other reasons, like the presence of bends, contractions, expansions, flow past submerged bodies, etc. and has to be included in the energy equation when such a loss is encountered.

Hydraulic and Energy Grade Lines

The pressure head p/γ can be determined at each section along a conduit running full by fitting piezometer tubes or pressure gauges. If piezometer tubes are used, the liquid will rise in them. The height of the top of the liquid column measured from the center of the conduit shall represent the pressure head p/γ and from the datum it will represent the sum of the elevation head and pressure head, i.e. $z + p/\gamma$, which is known as the *piezmetric head*. The line joining the piezometric heads is known as the hydraulic grade line (briefly written as H.G.L.).

Similarly the sum of the three heads, i.e. $z + p/\gamma + V^2/2g$, represents the total energy or head H at a section. The line joining the total energy or head is known as the total energy line (briefly written as T.E.L.). It is also known as the total head line or energy grade line. The total head is indicated by a Pitot tube, i.e. the line joining the liquid levels in the Pitot tube is the total energy line.

The total energy line lies above the hydraulic grade line and the distance between these two lines is the velocity head $\alpha V^2/2g$.

Head and Power

Each term of the Bernoulli or energy equation is called a head and represents the energy per unit weight having units of m-kg/kg or m-N/N or simply m. Power is defined as the rate of doing work. It is designated by P and obtained by multiplying the head H by the weight of liquid per second, i.e.

$$P = WH = \gamma QH \qquad (\text{in m-kg/s or m-N/s or W}) \qquad (6.27)$$

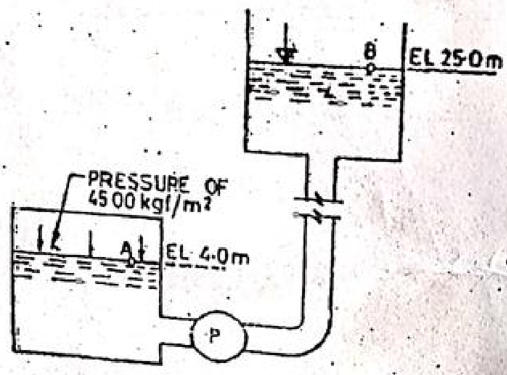
Horse power is a measure of power in MKS units and is equal to 75 m-kg/s.

$$\therefore HP = \frac{\gamma QH}{75} \qquad (6.28)$$

In SI units, power is measured in watt (W), which is equal to one joule per second.

Example 6.6

A 15 HP pump working with 80% efficiency is discharging crude oil (sp. gr. = 0.90) to the overhead tank shown in the figure. If losses in the whole system is 1.5 m of flowing fluid, find the discharge.



Solution Applying the energy equation between A and B

$$z_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} + H_{pump} = z_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + h_{l\,A-B}$$

where H_{pump} is the head supplied by the pump to the flow and $h_{l\,A-B}$ is the losses between A and B.

Here, $p_B/\gamma = 0$ since the pressure at B is atmospheric. Further, $V_A^2/2g$ and $V_B^2/2g$ are negligible and $h_{l\,A-B} = 1.5$ m.

$$\therefore 4 + \frac{4500}{0.9 \times 1000} + 0 + H_{pump} = 25 + 0 + 0 + 1.5$$

or

$$H_{pump} = 26.9 - 9 = 17.5 \text{ m of crude oil}$$

Now, HP of pump = 15 and efficiency of the pump = 80%

$\therefore$ Energy supplied by the pump to the flow system = $15 \times 0.80 = 12$ HP

$$\text{Again,} \quad HP = \frac{\gamma Q H_{pump}}{75}$$

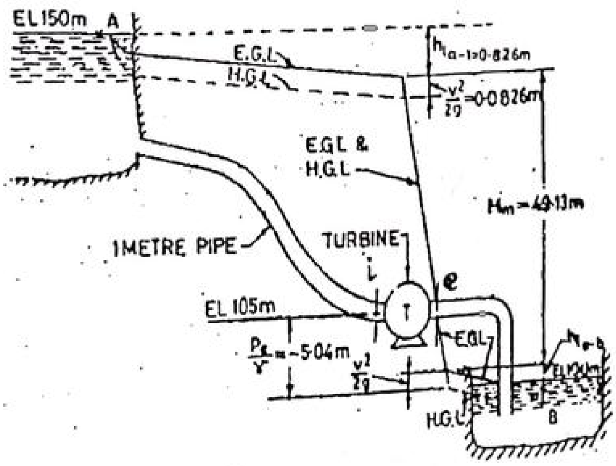
or

$$12 = \frac{900 \times Q \times 17.5}{75}$$

$$\therefore Q = \frac{75 \times 12}{900 \times 17.5} = 0.0571 \; m^3/s = 57.1 \; liters/sec$$

Example 6.7

A turbine T draws water from a reservoir through a 1 m diameter pipe and discharges through another pipe of the same diameter into tailrace B. The head loss from the headrace A to the turbine is found to be 10 times the velocity head in the pipe and that from the turbine to the tailrace B is only 0.5 time. If the discharge is 1 cumec, (a) calculate the pressure heads at inlet and exit of the turbine, (b) compute the power given up by the water to the turbine in HP and kW, and (c) draw the hydraulic grade line and the energy grade line.



Solution Diameter of the pipe, d = 1 m

∴ Area of the pipe,

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 1^2 = 0.785\ m^2$$

∴ Velocity of flow in the pipe, $V = \frac{Q}{A} = \frac{1}{0.785} = 1.273\ m/s$

$$\therefore \frac{V^2}{2g} = \frac{1.273^2}{2\times 9.81} = 0.0826\ m/s$$

(a) Applying the energy equation between A and i, the inlet to the turbine

$$z_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} = z_i + \frac{p_i}{\gamma} + \frac{V_i^2}{2g} + h_{lA-i}$$

where $h_{l\,A-i}$ is the head loss between A and i. Water in the reservoir is taken stationary. So, $V_A = 0$. Also, the pressure at the water surface in the reservoir is atmospheric. So, $p_A = 0$.

$$\therefore 150 + 0 + 0 = 105 + \frac{p_i}{\gamma} + 0.0826 + 10\times 0.0826$$

or

$$\frac{p_i}{\gamma} = 150 - 105 - 0.9086 = 44.0914 \text{ m of water}$$

Applying the energy equation between e, the exit of the turbine, and B

$$z_e + \frac{p_e}{\gamma} + \frac{V_e^2}{2g} = z_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + h_{le-B}$$

where $h_{l\,e-B}$ is the head loss between e and B. In the tailrace water is taken stationary. So, $V_B = 0$. Also, the pressure at the tailrace water surface is atmospheric. So, $p_B = 0$

$$\therefore 105 + \frac{p_e}{\gamma} + 0.0826 = 100 + 0 + 0 + 0.5\times 0.0826$$

or

$$\frac{p_e}{\gamma} = 100.0413 - 105.0826 = -5.0413 \text{ m of water}$$

(b) Applying the energy equation between A and B

$$z_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} - H_m = z_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + h_{lA-B}$$

where $h_{l\,A\text{-}B}$ is the total head loss in the system being equal to the sum of the losses from A to i and from e to B. Since the turbine T exists between A and B, a head H_m is extracted by it from the system.

$$\therefore\ 150 + 0 + 0 - H_m = 100 + 0 + 0 + 10.5 \times 0.0826$$

or

$$H_m = 150 - 100.8673 = 49.1327 \text{ m of water}$$

$$\therefore\ \text{Power} = \frac{\gamma Q H_m}{75} = \frac{1000 \times 1 \times 49.1327}{75} = 655\ HP \text{ (in MKS units)}$$

$$= \gamma Q H_m = 9.81 \times 1000 \times 1 \times 49.1327\ watt = 482\text{ kW (in SI units)}$$

(c) The hydraulic grade line (HGL) and the energy grade line (EGL) have been drawn in the figure which are self-explanatory. Note that p_e/γ is negative, so HGL is below the center line of pipe.

6.4 MOMENTUM EQUATION

Linear Momentum Equation

The fundamental principle of dynamics is the Newton's second law of motion which states that "The time rate of change of momentum is proportional to the applied force and takes place in the direction of force." More precisely, this statement may be written as "The time rate of change of linear momentum of a particle of mass m along any direction x is equal to the resultant external force F_x acting on a particle in the same direction x."

Momentum of a body is the product of its mass and velocity. Let the change in velocity in moving fluid mass m in time dt be dV. Since the velocity has changed, the momentum will also change.

$$\therefore \text{ Change of momentum} = m.dV$$

and

$$\text{Time rate of change of momentum} = m\frac{dV}{dt}$$

Then, according to the Newton's second law of motion, dynamic force applied in the x-direction = Rate of change of momentum in the x-direction, i.e.

$$F_x = m\frac{dV_x}{dt} \tag{6.29}$$

where the suffix x denotes the component in the x-direction. This equation is known as the *linear momentum equation* and can be written as

$$F_x\,dt = m\,dV_x \tag{6.30}$$

The left hand side of this equation is the product of force and the time increment during which it acts. This is known as the *impulse* of applied force. The right hand side is the resulting change in momentum. Equation (6.30) is known as the *impulse-momentum equation* which states that "The impulse is equal to the resulting change in the momentum of the body."

Figure 6.8 shows a steady flow of any fluid through a conduit. Consider the fluid mass enclosed between sections 1 and 2 as a free body. If F_x, F_y and F_z are the components of the resultant force F and V_x, V_y and V_z are the components of the velocity in the x, y and z directions, respectively, then

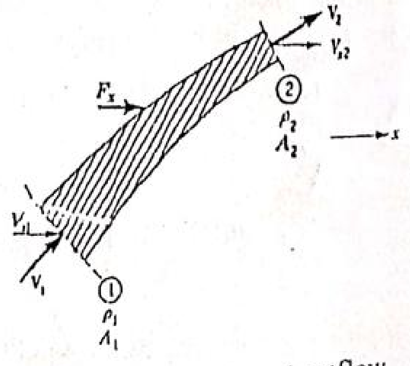


Fig. 6.8 Control volume with inflow and outflow

F_x = Change of momentum in the x-direction
= mass × change of velocity in the x-direction

$$= \rho Q(V_{x2} - V_{x1}) \quad (6.31a)$$

$$F_y = \rho Q(V_{y2} - V_{y1}) \quad (6.31b)$$

$$F_z = \rho Q(V_{z2} - V_{z1}) \quad (6.31c)$$

as the mass per second entering and leaving is $\rho Q = \rho_1 Q_1 = \rho_2 Q_2$.

The resultant force acting on the fluid body is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \quad (6.32)$$

For a two-dimensional flow in the x-y plane, $F_z = 0$ and we get

$$F = \sqrt{F_x^2 + F_y^2} \quad (6.33)$$

and if the resultant force F makes an angle θ with the horizontal, then

$$\theta = \tan^{-1}\frac{F_y}{F_x} \quad (6.34)$$

Momentum Coefficient

In the one-dimensional method of flow analysis, owing to non-uniform velocity distribution in a cross-section, the momentum of flow computed from the cross-sectional mean velocity is generally less than its actual value. To get the actual momentum, the mean velocity is multiplied by the coefficient β, known as the momentum based on the mean velocity is multiplied by the coefficient β, known as the *momentum correction factor* or the *Boussinesq coefficient*, *momentum coefficient* or the *momentum correction factor* or the *Boussinesq coefficient*, which is analogous to the energy coefficient α (Art. 6.3.4).

Referring to Fig. 6.1, the momentum of flow passing ΔA per unit time is $\rho v \Delta A \times v = \rho v^2 \Delta A$. Therefore, the total momentum of flow passing the cross-section per unit time is equal to

$$\rho \int_0^A v^2 dA$$

The total momentum based on the mean velocity V and corrected for the non-uniform distribution of velocity is $\beta \rho V^2 A$ so that

$$\beta = \frac{\int_0^A v^2 dA}{V^2 A} \quad (6.35)$$

The momentum coefficient, like the energy coefficient, is always positive and never less than unity. For uniform velocity distribution in the channel section, α = β = 1. In all other cases, α > β > 1. For turbulent flow, the numerical value of β does not normally exceed 1.04 and one can assume β = 1 without any appreciable error. For laminar flow in a straight circular pipe, β = 4/3.

When the momentum coefficient β is incorporated, the momentum or the impulse momentum equation should be modified and written, for example, as

$$F_x = \rho Q(\beta_2 V_{x2} - \beta_1 V_{x1}) \quad (6.36)$$

Example 6.8

For the velocity distribution given in Example 6.4, determine the momentum coefficient β.

Solution We have as in Example 6.4, the mean velocity, $V = \frac{98}{120} v_{max}$

$$\therefore \beta = \frac{1}{AV^2}\int_0^A v^2 dA = \frac{1}{\pi r_0^2} \times \left(\frac{120}{98}\right)^2 \times \frac{1}{v_{max}^2}\int_0^{r_0} v_{max}^2 \left(\frac{r_0 - y}{r_0}\right)^{2/7} \times 2\pi y dy$$

$$= \frac{2}{r_0^2} \times \left(\frac{120}{98}\right)^2 \times \frac{1}{r_0^{2/7}} \times \int_0^{r_0} (r_0 y^{2/7} - y^{9/7})\, dy = \frac{2}{r_0^2} \times \left(\frac{120}{98}\right)^2 \times \frac{1}{r_0^{2/7}}\left(\frac{7r_0^{16/7}}{9} - \frac{7r_0^{16/7}}{16}\right)$$

$$= 2 \times \left(\frac{120}{98}\right)^2 \times \frac{49}{14} = 1.020$$

Example 6.9

Derive the expression for the normal force when a jet of water strikes a stationary flat plate.

Solution When a jet of water with a velocity V strikes a stationary flat plate normally (Fig. 6.9), the force on the plate is equal to the rate of change of momentum of the jet. The jet leaves the plate tangentially so that all its momentum in a direction normal to the plate is destroyed. Hence, the normal force on the plate is

$$F = \rho Q(V - 0) = \rho QV = \rho AV^2$$

where A is the cross-sectional area of the jet.

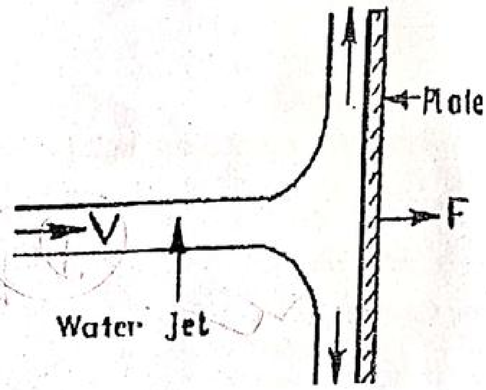


Fig. 6.9 Jet striking a flat plate

PROBLEMS AND EXERCISES

6.1 Derive the equation of continuity for (i) one-dimensional steady flow, and (ii) three-dimensional unsteady flow.

6.2 What are the three forms of energy present in fluid flow? Why the term 'head' is used for them? Why do you mean by frictional loss of head?

6.3 State the Bernoulli's theorem and write it mathematically. What are the limitations of this theorem?

6.4 Derive the Euler equation and state the assumptions of this equation.

6.5 Why are the kinetic energy and the momentum coefficients used? Derive expressions for them.

6.6 What do you mean by HGL and TEL?

6.7 A pipe AB branches into two pipes C and D as shown in Fig. 6.10. The pipe has diameter of 45 cm at A, 30 cm at B, 20 cm at C and 25 cm at D. Determine the discharge at A, if the velocity at A is 2 m/s. Also, determine the velocity at B and D, if the velocity at C is 4 m/s.

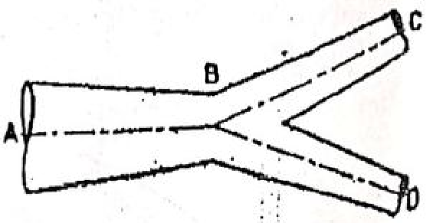


Fig. 6.10 (Problem 6.7)

6.8 The velocity distribution for a two-dimensional incompressible flow is given by

$$u = -\frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2}$$

Show that it satisfies the equation of continuity.

6.9 Given that $u = 2x^2 + 2xy$ and $v = 2yz^2 + 3z^2$, find the component of velocity in the z-direction so that the equation of continuity for an incompressible flow is satisfied.

6.10 A pipe 300 m long has a slope of 1 in 100 and tapers from 1 m diameter at the higher end to 0.5 m at the lower end, as shown in Fig. 6.11. Quantity of water flowing is 5400 liters/minute. If the pressure at the higher end is 0.7 kg/m^2, find the pressure at the lower end. Neglect losses.

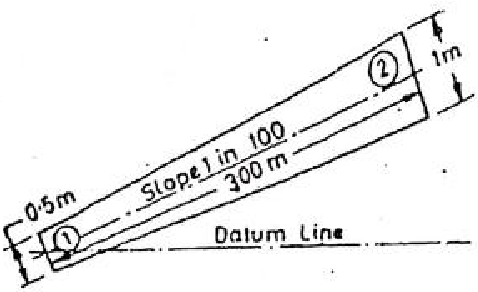


Fig. 6.11 (Problem 6.10)

6.11 In a pipe the velocity varies linearly from 0 at the pipe wall to maximum at the center of the pipe. Determine the kinetic energy coefficient α and momentum coefficient β.

6.12 Figure 6.12 shows a sharp-crested weir in a rectangular channel. If the discharge per unit width of the weir is 4 m^2/s, estimate the energy loss due to the weir and force on the weir plate for the submerged flow condition as shown.

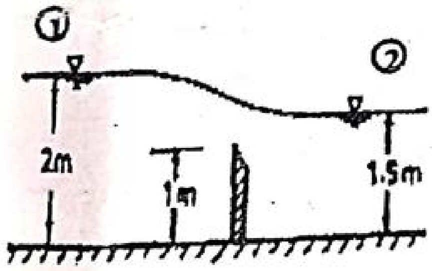


Fig. 6.12 (Problem 6.12)

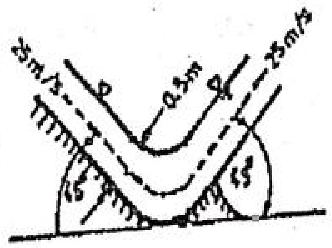


Fig. 6.13 (Problem 6.13)

6.13 The inlet and exit angles of a ski-jump spillway (Fig.6.13) are 45° and the flow over it has a velocity of 25 m/s and a depth of 0.5 m. Neglecting all losses, estimate the maximum elevation of the outflow trajectory. Also, compute the horizontal and vertical forces on the spillway as a result of the change in the flow direction. Assume unit width.

6.14 In Example 6.9, if the jet of water strikes the plate normally and leaves the plate normally, what would be the normal force on the plate?

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Chapter 7

DIMENSIONAL ANALYSIS AND SIMILITUDE

7.1 INTRODUCTION

In modern hydraulic engineering, several new problems are met by the engineers which generally pertain to the design, construction and efficient working of various types of hydraulic structures and machines. Some of these problems may be easily solved by the mathematical analysis. However, there are some problems for which a purely theoretical investigation fails to yield a practical and workable solution due to their complex nature, and it is thus necessary to rely on experimental results. The solution of such complex problems is considerably simplified by the use of dimensional analysis.

It is not possible to carry out experiments on full size dams, rivers, channels or hydraulic machines such as turbines and pumps because it is very costly. For the sake of economy and convenience, it is essential that small scale models be made for the prototypes for the purpose of testing. So, certain laws of similarity must be observed in order to ensure that model results can be applied to the prototype.

7.2 DIMENSIONAL ANALYSIS

Dimensional analysis is a mathematical technique which deals with the dimensions of physical quantities involved in the phenomenon. Each physical phenomenon can be expressed by an equation giving relationship between different quantities. In general, any variable present in a physical phenomenon is a dimensional quantity. Dimensional analysis helps us to determine a convenient arrangement of quantities or variables in a physical relationship. This is accomplished by forming a number of non-dimensional groups out of a given number of dimensional quantities so that variables can be reduced. These days the dimensional analysis is widely used in research work for developing design criteria and also for conducting model test.

The use of dimensional analysis can be summarized as follows:

1. It helps to find whether an equation of any flow phenomenon is rational or not. The equation is called *rational* if it is dimensionally homogeneous. This is to say that dimensional analysis can be applied only when a phenomenon can be expressed by a dimensionally homogeneous equation.

2. The relationship between various physical quantities in an equation, governing a particular flow phenomenon, can be known.

3. It helps the reduction of a number of variables involved in a flow phenomenon with which the performance of experiments becomes easy.

4. Rational formula for the flow phenomenon can be derived.

5. It helps in making suitable smaller-sized models in which experiments can be performed to predict the performance of prototypes.

6. Dimensional analysis, on the whole, facilitates planning of reliable scientific experiments.

7.3 FUNDAMENTAL AND DERIVED QUANTITIES

In the MLT system, there are only three fundamental quantities mass, length and time and they are represented by the letters M, L and T, respectively. All other quantities such as area, volume, velocity, acceleration, force, energy, power, etc. are called derived quantities, because they can be expressed in terms of the above fundamental quantities.

Some engineers prefer to use force instead of mass as fundamental quantity as the former is easy to measure. The system is then represented by FLT. Some of the derived quantities are given in Table 7.1 in terms of the fundamental quantities of both systems. In this lecture note, we use the MLT system.

Table 7.1 Dimensions and units of some physical quantities

Sl. No.	Quantity & Symbol	SI unit	Dimension in MLT system	Dimension in FLT system
1.	Length (l)	m	L	L
2.	Area (A)	m^2	L^2	L^2
3.	Volume (V)	m^3	L^3	L^3
4.	Time (t)	s	T	T
5.	Velocity (V)	m/s	LT^{-1}	LT^{-1}
6.	Acceleration (a)	m/s^2	LT^{-2}	LT^{-2}
7.	Gravitational acceleration (g)	m/s^2	LT^{-2}	LT^{-2}
8.	Frequency (N)	/s	T^{-1}	T^{-1}
9.	Discharge (Q)	m^3/s	L^3T^{-1}	L^3T^{-1}
10.	Force (F)/Weight (W)	N	MLT^{-2}	F
11.	Power (P)	W	ML^2T^{-1}	FLT^{-1}
12.	Work/Energy (E)	N-m or J	ML^2T^{-2}	FL
13.	Pressure (p)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
14.	Mass (M)	kg	M	FT^2L^{-1}
15.	Mass density (ρ)	kg/m^3	ML^{-3}	FT^2L^{-4}
16.	Specific weight (γ)	N/m^3	$ML^{-2}T^{-2}$	FL^{-3}
17.	Dynamic viscosity (μ)	$N\text{-}s/m^2$	$ML^{-1}T^{-1}$	FTL^{-2}
18.	Kinematic viscosity (ν)	m^2/s	L^2T^{-1}	L^2T^{-1}
19.	Surface tension (σ)	N/m	MT^{-2}	FL^{-1}
20.	Shear stress (τ)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
21.	Modulus of elasticity (K)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
22.	Angular velocity (ω)	rad/s	T^{-1}	T^{-1}
23.	Torque (T)	N-m	ML^2T^{-2}	FL

7.4 DIMENSIONAL HOMOGENEITY

An equation is called dimensionally homogeneous if the dimensions have identical powers on both sides. Such an equation is also known as a rational equation. A dimensionally homogeneous equation would essentially be independent of the system of units (i.e. SI, Metric or English). Let us consider the most common equation of hydraulics

$$Q = AV$$

Dimensionally,

$$L^3T^{-1} = L^2 \times LT^{-1}$$
$$= L^3T^{-1}$$

So, both sides of the equation have the same dimensions. Therefore, the equation is dimensionally homogeneous or rational.

The principle of dimensional homogeneity can be used (i) to determine the dimension of a physical quantity, (ii) to check the dimensional homogeneity of an equation, and (iii) to change the coefficient of an equation while using it in other systems of units.

7.5 METHODS OF DIMENSIONAL ANALYSIS

Dimensional analysis enables the mathematical formulation of a physical phenomenon when the variables which are involved in the phenomenon are known. The methods of dimensional analysis enable us to combine the variables involved into compact non-dimensional groups and obtain a functional relationship between them. There are two methods of dimensional analysis:

1. Rayleigh's method
2. Buckingham's method, generally known as Buckingham's π-theorem.

Rayleigh's Method

This method was originally proposed by Lord Rayleigh in 1899. In this method, the dependent variable is expressed as a function of the exponents (or powers) of the independent variables. The dependent variable is one for which information is required, whereas the independent variables are those which govern the variation of the dependent variable. If y is the dependent variable and x_1, x_2, x_3, ... are the independent variables involved in a physical process, then the functional relation may be written as

$$y = f(x_1^a, x_2^b, x_3^c, ...)$$

where a, b, c, ...are the exponents or powers of x_1, x_2, x_3, ..., respectively The values of a, b, c, ... are obtained using the principle of dimensional homogeneity.

Example 7.1

If the capillary rise (h) depends upon the specific weight (γ), surface tension of the liquid (σ) and the radius of the tube (r), show that

$$h = r f\left(\frac{\sigma}{\gamma r^2}\right)$$

Solution Let the functional relationship be

$$h \propto \gamma^a . \sigma^b . r^c \qquad \text{(i)}$$

Substituting the respective dimensions in (i), we get

$$L \propto (ML^{-2}T^{-2})^a . (MT^{-2})^b . (L)^c \qquad \text{(ii)}$$

Now, for dimensional homogeneity, equating the powers of M, L and T on both sides of (ii), we get

for M, $0 = a + b$ (iii)
for L, $1 = -2a + c$ (iv)
for T, $0 = -2a - 2b$ (v)

From (iii), we get $a = -b$
From (iv), we get $c = 1 + 2a = 1 - 2b$

Now, substituting the values of a and c in (i), we get

$$h \propto \gamma^{-b} . \sigma^b . r^{1-2b}$$

$$h \propto \frac{1}{\gamma^{b}} \times \sigma^{b} \times \frac{r}{r^{2b}}$$

or, $h \propto r\left(\dfrac{\sigma}{\gamma r^2}\right)^{b}$ or, $\dfrac{h}{r} \propto \left(\dfrac{\sigma}{\gamma r^2}\right)^{b}$

∴ The functional relationship may be expressed as

$$\frac{h}{r} = f\left(\frac{\sigma}{\gamma r^2}\right) \quad \text{or,} \quad h = r f\left(\frac{\sigma}{\gamma r^2}\right)$$

Example 7.2

Derive a rational formula for pipe flow having given the following quantities affecting the flow phenomenon: F is the boundary friction force, μ is the viscosity, ρ is the mass density, V is the velocity of flow, D is the pipe diameter and k is the surface roughness.

Solution Assuming that $F = f(\mu, \rho, V, D, k)$ (i)

$$F \propto \mu^a . \rho^b . V^c . D^d . k^e \quad \text{(ii)}$$

or, dimensionally

$$MLT^{-2} \propto (ML^{-1}T^{-1})^a . (ML^{-3})^b . (LT^{-1})^c . (L)^d . (L)^e \quad \text{(iii)}$$

Equating indices of M, L and T on both sides, we get

For M, $1 = a + b$ (iv)

For L, $1 = -a - 3b + c + d + e$ (v)

For T, $-2 = -a - c$ (vi)

From (iv), we have $b = 1 - a$ (vii)

From (vi), we have $c = 2 - a$ (viii)

From (v), we have $d = 1 + a + 3b - c - e = 1 + a + 3 - 3a - 2 + a - e$ (substituting for b and c) (ix)

$= 2 - a - e$

Eliminating b, c and d from (ii), we have

$$F \propto (\mu)^a . (\rho)^{1-a} . (V)^{2-a} (D)^{2-a-e} . (k)^e$$

or

$$F \propto \rho V^2 D^2 \left(\frac{\rho V D}{\mu}\right)^{-a} . (k/D)^e$$

or

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}, \frac{k}{D}\right)$$

The non-dimensional quantity ρVD/μ is the Reynolds number, Re, which is the ratio between inertia and viscous forces and k/D is the relative roughness. Thus, the above equation becomes a rational equation.

Buckingham's Method or Buckingham's π-Theorem

This method was originally proposed by Edgar Buckingham in 1915. The Rayleigh's method of dimensional analysis becomes cumbersome when a large number of variables are involved. In order to overcome this difficulty, the Buckingham's method may be conveniently used. It states "*If there are n variables in a dimensionally homogeneous equation, and if these variables contain m fundamental dimensions, they may be grouped into n-m non-dimensional independent parameters.*"

Mathematically, if a variable x_1 depends upon a number of independent variables x_2, x_3, x_4, …, x_n, the functional equation may be written as

$$x_1 = \phi(x_2, x_2, x_3, \ldots, x_n)$$

where ϕ is a function. This equation may be written in its general form as

$$f(x_1, x_2, x_2, x_3, \ldots, x_n) = 0$$

where f is another function.

If $\pi_1, \pi_2, \pi_3, \ldots$ represent dimensionless groupings of the variables x_1, x_2, x_3, ……, x_n with m dimensions involved, then according to Buckingham's π-theorem, an equation of the form

$$F(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-m}) = 0 \qquad (7.1)$$

exists, where F is still another function.

The steps of the Buckingham's method are as follows:

1. Write the functional relationship with the given data.
2. Then write the equation in its general form.
3. Now choose m *repeating variables* and write separate expression for each π-term. Each π-term will contain the repeating variables and one of the remaining variables. The repeating variables are written in exponential form.
4. Use the principle of dimensional homogeneity to find out the values of the exponents or powers by obtaining simultaneous equations.
5. Now substitute the values of the exponents or powers in the π-terms.
6. After the π-terms are determined, write the functional relation in the required form.

The following points must be considered while selecting the repeating variables:

1. The number of repeating variables should be equal to m, the number of fundamental dimensions.
2. The repeating variables should not be dimensionless and the repeating variables among themselves should not form a dimensionless number.
3. Independent variables should, as far as possible, be selected as repeating variables.

Generally, the repeating variables selected are ρ, V and l, i.e. the first repeating variable will represent the fluid property, the second will represent the flow characteristics and the third will represent the geometrical characteristics of the body.

Example 7.3

The pressure drop Δp in a pipe depends upon the mean velocity of flow (V), length of pipe (l), diameter of pipe (D), viscosity of fluid (μ), average height of roughness elements

(k) and mass density (ρ). (a) By using Buckingham's π-theorem, obtain a dimensionless expression for Δp. (b) Show that

$$h_f = \frac{4flV^2}{2gD}$$

where h_f is the head loss due to friction (Δp/γ) and γ is the specific weight of the fluid and f is the coefficient of friction.

Solution (a) Let the functional relationship be

$$\Delta p = f(V, l, D, \mu, k, \rho)$$

The above equation can be written in its general form as

$$f_1(\Delta p, V, l, D, \mu, k, \rho) = 0$$

A little consideration shows that in the above equation the fundamental dimensions are three. Thus, m = 3. Since n = 7, so there will be n − m = 7 − 3 = 4 π-parameters. Taking V, D and ρ as repeating variables, we get

$$\pi_1 = V^{a_1}.D^{b_1}.\rho^{c_1}.\Delta p$$

or

$$\pi_1 = (LT^{-1})^{a_1} \times (L)^{b_1} \times (ML^{-3})^{c_1} \times (ML^{-1}T^{-2}) = M^0L^0T^0$$

For M, $0 = c_1 + 1$ or, $c_1 = -1$
For T, $0 = a_1 - 2$ or, $a_1 = 2$
For L, $0 = a_1 + b_1 - 2c_1$ or, $b_1 = 0$

$$\therefore \pi_1 = V^{-2}.D^0.\rho^{-1}.\Delta p = \frac{\Delta p}{\rho V^2} \quad \text{(i)}$$

Similarly,

$$\pi_2 = V^{a_2}.D^{b_2}.\rho^{c_2}.l$$

or

$$M^0L^0T^0 = (LT^{-1})^{a_2} \times (L)^{b_2} \times (ML^{-3})^{c_2} \times (L)$$

For M, $0 = c_2$ or, $c_2 = 0$
For T, $0 = -a_2$ or, $a_2 = 0$
For L, $0 = a_2 + b_2 - 3c_2 + 1$ or, $b_2 = -1$

$$\pi_2 = V^0.D^{-1}.\rho^0.l = \frac{l}{D} \quad \text{(ii)}$$

Similarly,

$$\pi_3 = V^{a_3}.D^{b_3}.\rho^{c_3}.\mu$$

or

$$M^0L^0T^0 = (LT^{-1})^{a_3} \times (L)^{b_3} \times (ML^{-3})^{c_3} \times (ML^{-1}T^{-1})$$

For M, $0 = c_3 + 1$ or, $c_3 = -1$
For T, $0 = -a_3 - 1$ or, $a_3 = -1$
For L, $0 = a_3 + b_3 - 3c_3 - 1$ or, $b_3 = -1$

$$\pi_3 = V^{-1}.D^{-1}.\rho^{-1}.\mu = \frac{\mu}{VD\rho} \quad \text{(iii)}$$

Lastly,

$$\pi_4 = V^{a_4}.D^{b_4}.\rho^{c_4}.k$$

or

$$M^0L^0T^0 = (LT^{-1})^{a_4} \times (L)^{b_4} \times (ML^{-3})^{c_4} \times (L)$$

For M,	$0 = c_4$	or,	$c_4 = 0$
For T,	$0 = -a_4$	or,	$a_4 = 0$
For L,	$0 = a_4 + b_4 - 3c_4 + 1$	or,	$b_4 = -1$

$$\pi_4 = V^0.D^{-1}.\rho^0.k = \frac{k}{D} \qquad \text{(iv)}$$

Now, the functional relationship can be written as

$$F_1(\frac{\Delta p}{\rho V^2}, \frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D}) = 0 \qquad \text{(v)}$$

or, $$F_2(\frac{\Delta p}{\rho V^2} \times \frac{D}{l}, \frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D}) = 0$$

or, $$F_2(\frac{\Delta pD}{\rho V^2 l}, \frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D}) = 0$$

or, $$\frac{\Delta pD}{\rho V^2 l} = F_3(\frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D})$$

or, $$\Delta p = \frac{\rho V^2 l}{D} F_3(\frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D}) \qquad \text{(vi)}$$

Equation (vi) is the required relationship.

(b) Let $$2f = F_3(\frac{l}{D}, \frac{\mu}{VD\rho}, \frac{k}{D})$$

$$\therefore \Delta p = \frac{\rho V^2 l}{D}.2f = \frac{4f\rho V^2 l}{2D}$$

Dividing the left hand side by the specific weight of the fluid γ and the right hand side by its equivalent ρg, we get

$$\frac{\Delta p}{\gamma} = \frac{4f\rho V^2 l}{2\rho g D} \qquad \text{or,} \qquad h_f = \frac{4flV^2}{2gD} \quad \text{(Proved)}$$

7.6 MODEL TESTING

Since the beginning of the twentieth century, the engineers have started a new and scientific method to predict the performance of their structures and machines before they are made and installed. This is done by preparing models and testing them in the laboratory, so as to form some opinions about the working and behavior of the proposed structures, after their completion or actual installation. The structure, of which the model is prepared, is known as *prototype* and the model is known as the *physical model* or *scale model* or simply *model*.

The principal advantages of model testing are:

1. The behavior and working details of a hydraulic structure or machine can be easily predicted by performing experiments on its model.

2. If the hydraulic structure or machine is made directly, then in case of failure, it is very difficult to change its design. Moreover, it is very costly to build the same. Thus, making a model and testing it in the laboratory results in the saving of human labor and material.

3. With the help of model testing, a number of alternative designs can be studied. Finally the most economical, accurate and safe design may be selected.

4. In case when an existing structure is not functioning properly, its defects can be detected and rectified by model testing.

5. Sometimes, it is very difficult to design a particular portion of a complex structure or machine. In such cases the model testing becomes absolutely necessary to ascertain the safety and reliability of that particular portion of the prototype.

Following are the fields where the application of the model testing is of great use:

(a) Water power engineering
- (i) Turbines and pumps
- (ii) Hydraulic structures like dams, weirs, spillways, canal falls, etc.

(b) Irrigation engineering
- (i) Flood control
- (ii) River training
- (iii) Investigation of silting and scouring in rivers, irrigation canals, etc.

(c) Design of ships, harbors, etc.

(d) Design of aeroplanes for compressible flow.

7.7 HYDRAULIC SIMILARITY OR SIMILITUDE

In order to have a complete idea about the construction and working of a hydraulic structure or machine, it is very essential that the model should represent its prototype completely and fully in all respects. The similarity between the prototype and its model is known as hydraulic similarity or hydraulic similitude.

There are three kinds of similarities which a model should possess in order that it may reproduce the behavior of its prototype: (i) geometric similarity, (ii) kinematic similarity, and (iii) dynamic similarity.

Geometric Similarity

It is similarity of shape between the model and the prototype. The model must be an exact replica of the prototype, i.e. identical in shape but different in size. The geometric similarity is said to exist between the model and the prototype if the ratios of all the corresponding linear dimensions are equal and all the identical angles are same in both cases.

Let L_p, B_p and D_p are the length, width and depth of the prototype and L_m, B_m and D_m are the corresponding values for the model. The *scale ratio* (L_r) is the ratio of the linear dimension of the model to that of the prototype. If the geometric similarity exists between the model and the prototype, then

$$L_r = \frac{L_m}{L_p} = \frac{B_m}{B_p} = \frac{D_m}{D_p} \quad (7.2)$$

Similarly, the area ratio (A_r) between the model and the prototype

$$A_r = \left(\frac{L_m}{L_p}\right)^2 = \left(\frac{B_m}{B_p}\right)^2 = \left(\frac{D_m}{D_p}\right)^2 \quad (7.3)$$

and the volume ratio (V_r) between the model and the prototype

$$V_r = \left(\frac{L_m}{L_p}\right)^3 = \left(\frac{B_m}{B_p}\right)^3 = \left(\frac{D_m}{D_p}\right)^3 \quad (7.4)$$

Kinematic Similarity

It is the similarity of motion. The kinematic similarity is said to exist between the model and the prototype if the ratio of the corresponding velocities of the fluid particles at the corresponding points in the model and the prototype is the same. Similarly, the ratio of accelerations of the fluid particles in both cases should also be the same. Further, the direction of flow at the corresponding points should be the same.

Let V_{1p} and V_{2p} are the velocities of the fluid particles in the prototype at points 1 and 2, respectively, and V_{1m} and V_{2m} are the corresponding values for the model. Now, if the kinematic similarity exists between the model and the prototype, then the velocity ratio (V_r) of the model and the prototype

$$V_r = \frac{V_{1m}}{V_{1p}} = \frac{V_{2m}}{V_{2p}} = \frac{V_{3m}}{V_{3p}} = \ldots\ldots \quad (7.5)$$

Dynamic Similarity

It is the similarity of forces. But the geometric and kinematic similarities are necessary for dynamic similarity. The dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points in the model and the prototype are equal.

Let F_{1p} and F_{2p} are the forces acting in the prototype at points 1 and 2, respectively, and F_{1m} and F_{2m} are the corresponding values for the model. Now, if the dynamic similarity exists between the model and the prototype, then the force ratio (F_r) of the model and the prototype

$$F_r = \frac{F_{1m}}{F_{1p}} = \frac{F_{2m}}{F_{2p}} = \frac{F_{3m}}{F_{3p}} = \ldots\ldots \quad (7.6)$$

A model which satisfies all the above conditions is known as a *completely similar and true model*. In practice, however, it is not possible to achieve complete similarity in a model. It is, therefore, common to consider only those forces which are predominant in a phenomenon and design the model such that the same forces influence the flow phenomenon in the model also. The effect of other forces which are insignificant is either neglected or taken care of by introducing correction factors based on experiments or experience.

7.8 DIMENSIONLESS OR NON-DIMENSIONAL NUMBERS

When a fluid is in motion, some forces are always involved in the phenomenon of flow. But there are always one or two forces, which dominate other forces, and they govern the flow of the fluid and keep it in motion. The following forces are important in fluid motion:

1. Inertia force
2. Friction or viscous force
3. Gravity force
4. Surface tension force
5. Pressure force
6. Elastic force

Since the inertia force always exists in the phenomenon of fluid flow, the conditions of dynamic similarity are always studied considering the ratio of the inertia force and any of the remaining forces. Obviously, every ratio will be a dimensionless or non-dimensional number. Some of the most important numbers are (i) Reynolds number, (ii) Froude number, (iii) Weber number, (iv) Euler number, and (v) Mach number.

Define

Reynolds Number

The Reynolds number (Re) is the ratio of inertia force and viscous force, i.e.

$$Re = \frac{Inertia\ force}{Viscous\ force} = \frac{Mass \times acceleration}{Shear\ stress \times area} = \frac{M \times L/T^2}{\tau \times A} = \frac{\rho L^3 \times L/T^2}{\mu \frac{dv}{dz} \times A}$$

$$= \frac{\rho L^4/T^2}{\mu \times \frac{L/T}{L} \times L^2} = \frac{\rho L^2/T}{\mu} = \frac{(L/T).L}{\mu/\rho} = \frac{VL}{\nu} \quad (\because \nu = \mu/\rho) \qquad (7.7)$$

where V = a characteristic velocity, L = a characteristic length and ν is the kinematic viscosity of the fluid.

The non-dimensional ratio VL/ν is named as Reynolds number (Re) in the honour of Osborne Reynolds (1842-1912), a British scientist and mathematician. Reynolds number is a measure of the magnitude of the viscous force relative to the inertial force. The smaller is the Reynolds number the greater will be the relative magnitude of the viscous force and vice versa.

In a type of flow in which only the viscous force plays an important role relative to the inertia force and the effect of other forces is insignificant, dynamic similarity is said to exist between the model and the prototype when the Reynolds number for the model and the prototype is the same. Flow of an incompressible fluid in a pipe with a low velocity and groundwater flow are some of the examples where viscous force may be predominating.

Example 7.4

The performance of a ship was predicted by making its model and testing it in a wind tunnel. The length of the prototype was 350 m and its model was 10 m long. The ν for air is 1.25 times that of water. The velocity of air around the model in the wind tunnel was measured as 35 m/s. Find the velocity of actual ship in water if the model has dynamic similarity with the prototype and the flow is governed by Reynolds law.

Solution: Here, $L_p = 350$ m, $L_m = 10$ m, $\nu_m = 1.25\nu_p$, $V_m = 35$ m/s

We know, for dynamic similarity between the model and the prototype, the Reynolds number of model and prototype should be equal.

$$\therefore \frac{V_p L_p}{\nu_p} = \frac{V_m L_m}{\nu_m}$$

$$\therefore V_p = \frac{\nu_p}{\nu_m} \times \frac{L_m}{L_p} \times V_m = \frac{1}{1.25} \times \frac{10}{350} \times 35 = 0.8\ m/s$$

Froude Number

The Froude number (Fr) is the square root of the ratio of inertia force and gravity force, i.e.

$$Fr = \sqrt{\frac{Inertia\ force}{Gravity\ force}} = \sqrt{\frac{Mass \times Inertial\ acceleration}{Mass \times Gravitational\ acceleration}} = \sqrt{\frac{M \times a}{M \times g}} = \sqrt{\frac{a}{g}}$$

$$= \sqrt{\frac{L/T^2}{g}} = \sqrt{\frac{(L/T)^2}{gL}} = \sqrt{\frac{V^2}{gL}} = \frac{V}{\sqrt{gL}} \qquad (7.8)$$

where V = a characteristic velocity, L = a characteristic length and g (= 9.81 m/s^2) is the acceleration due to gravity.

The non-dimensional ratio $V/\sqrt{gL}$ is named as Froude number (Fr) in the honour of William Froude (1810-1889), a British scientist who first applied it to the practical problems of the resistance of ships and floating bodies. Froude number is a measure of the magnitude of the gravity force relative to the inertial force. The smaller is the Froude number the greater will be the relative magnitude of the gravity force and vice versa.

If the gravity force is the predominating force relative to the inertial force in the model and the prototype and the effect of other forces is negligible, dynamic similarity is said to exist between the two when the Froude number for the model and the prototype is the same. This is the case when the flow occurs with a free surface as in an open channel.

Example 7.5

A channel model 250 mm deep is discharging water with a velocity of 1.5 m/s. Find the velocity of water in the channel 4 m deep, if the model has dynamic similarity with its prototype and the flow is governed by Froude law.

Solution Depth of water in the model, $h_m = 250\ mm = 0.25\ m$

Depth of flow in the prototype, $h_p = 4\ m$

$$\therefore Scale\ ratio, L_r = \frac{h_m}{h_p} = \frac{0.25}{4} = \frac{1}{16} = \frac{L_m}{L_p}$$

$$\therefore \frac{L_m}{L_p} = \frac{1}{16} \qquad \text{or,} \qquad \frac{L_p}{L_m} = 16$$

$V_m = 1.5\ m/s$, $g_p = g_m$ (since the model and the prototype are in the same place)

We know, for dynamic similarity between the model and the prototype, the Froude number of model and prototype should be equal.

$$\therefore \frac{V_p}{\sqrt{g_p L_p}} = \frac{V_m}{\sqrt{g_m L_m}}$$

$$\therefore V_p = \sqrt{\frac{g_p}{g_m}} \times \sqrt{\frac{L_p}{L_m}} \times V_m = \sqrt{1} \times \sqrt{16} \times 1.5 = 6\ m/s$$

Weber Number

The Weber number (We) is the ratio of inertia force and surface tension force, i.e.

$$We = \frac{Inertia\ force}{Surface\ tension\ force} = \frac{Mass \times acceleration}{Surface\ tension\ force\ per\ unit\ length \times Length}$$

$$= \frac{\rho L^3 \times L/T^2}{\sigma.L} = \frac{\rho L^3/T^2}{\sigma} = \frac{\rho L(L/T)^2}{\sigma} = \frac{\rho L V^2}{\sigma} \qquad (7.9)$$

where σ is the force of surface per unit length.

The Weber number (We) is in the honor of Moritz Weber (1871-1951), Professor of Berlin. The force of surface tension becomes important when the diameter of a tube is small (i.e. a capillary tube) and when the depth of flow or the head is small (less than 5 mm). In these cases, dynamic similarity is said to exist between the model and the prototype when the Weber number for the model and the prototype is the same.

Euler Number

The Euler number (Eu) is the square root of the ratio of inertia force and pressure force, i.e.

$$Eu = \sqrt{\frac{Inertia\ force}{Pressure\ force}} = \sqrt{\frac{Mass \times Inertial\ acceleration}{Intensity\ of\ pressure \times Area}} = \sqrt{\frac{\rho L^3 \times L/T^2}{p \times L^2}}$$

$$= \sqrt{\frac{\rho L^2/T^2}{p}} = \sqrt{\frac{\rho (L/T)^2}{p}} = \sqrt{\frac{\rho V^2}{p}} = \frac{V}{\sqrt{p/\rho}} \quad (7.10)$$

The Euler number (Eu) is in the honour of Leonhard Euler (1707-1783). If the pressure force is the predominating force in the model and the prototype, dynamic similarity is said to exist between the two when the Euler number for the model and the prototype is the same. Practical applications of the Euler number are water hammer in penstocks of hydropower plants and discharge coefficients of orifices, mouthpieces and sluice gates, etc.

Example 7.6

A spillway model is to be built to a geometrically similar scale of 1/50 across a flume of 60 cm width. If the negative pressure in the model is 20 cm, what is the negative pressure in the prototype? Is it practicable?

Solution $L_r = 1/50$, $p_m = 20\ cm = 0.2\ m$

We know, for dynamic similarity between the model and the prototype, the Froude number of model and prototype should be equal.

$$\therefore \frac{V_p}{\sqrt{g_p L_p}} = \frac{V_m}{\sqrt{g_m L_m}}$$

or

$$\frac{V_p}{V_m} = \sqrt{\frac{g_p}{g_m}} \times \sqrt{\frac{L_p}{L_m}} = \sqrt{1} \times \sqrt{50} = \sqrt{50}$$

Now, in order to have dynamic similarity for pressure, the Euler number for the model and for the prototype should be equal, i.e.

$$\therefore \frac{V_p}{\sqrt{p_p/\rho_p}} = \frac{V_m}{\sqrt{p_m/\rho_m}}$$

or,

$$\sqrt{\frac{p_p}{p_m}} = \sqrt{\frac{\rho_p}{\rho_m}} \times \frac{V_p}{V_m} = \sqrt{1} \times \sqrt{50} = \sqrt{50}$$

$$\therefore \frac{p_p}{p_m} = 50 \qquad \therefore p_p = p_m \times 50 = 0.2 \times 50 = 10\ m$$

The negative pressure of 10 m as obtained is not practicable, as cavitation will occur at this pressure.

Mach Number

It is the square root of the ratio of inertia force and elastic force, i.e.

$$Ma = \sqrt{\frac{Inertia\ force}{Elastic\ force}} = \sqrt{\frac{Mass \times Inertial\ acceleration}{Elastic\ stress \times Area}} = \sqrt{\frac{\rho L^3 \times L/T^2}{K \times L^2}}$$

$$= \sqrt{\frac{\rho L^2/T^2}{K}} = \sqrt{\frac{\rho (L/T)^2}{K}} = \sqrt{\frac{\rho V^2}{K}} = \frac{V}{\sqrt{K/\rho}} \quad (7.11)$$

The Mach number is in the honor of Ernst Mach (1838-1916), Professor of Physics of Paraguay. In dealing with compressible fluids, elastic forces are important. The compressibility effect becomes significant when Mach number of the flow is greater than about 0.2. If the elastic force is the predominating force in the model and the prototype, dynamic similarity is said to exist between the two when the Mach number for the model and the prototype is the same.

Example 7.7

An airfoil moves at 650 km/hour through still air at 20^0C. If the elastic stress and density of air at this temperature are 21 kg/cm^2 and 0.126 kg/m^3, find the Mach number.

Solution Here, $V = 650 \text{ km/hour} = \dfrac{650 \times 1000}{60 \times 60} = 180.6 \text{ m/s}$

$$K = 21 \text{ kg/cm}^2 = 21 \times 10^4 \text{ kg/m}^2, \quad \rho = 0.126 \text{ kg/m}^3$$

$$\therefore Ma = \frac{V}{\sqrt{K/\rho}} = \frac{180.6}{\sqrt{21 \times 10^4 / 0.126}} = 0.14$$

7.9 TYPES OF MODELS

All the models may be broadly classified into the following two types:

1. Undistorted model
2. Distorted model

Undistorted Model: A model is said to be undistorted when the horizontal and vertical scale ratios (model distance: prototype distance) are same. An undistorted model is geometrically similar to its prototype. The prediction of an undistorted model is comparatively easy and the model results can be easily transferred to the prototype as the basic condition of geometric similarity is satisfied.

Distorted Model: A model is said to be distorted when the vertical and horizontal scale ratios are different. The distorted model does not have true or complete geometric similarity. For example, models of rivers, harbors, reservoirs, etc. have very large horizontal dimensions, as compared to vertical ones. If a model of such a prototype having a complete geometrical similarity is made, then the depth of water in such a model becomes so small that it cannot be accurately measured. To overcome this difficulty, the vertical scale of the model is increased relative to the horizontal scale.

The prediction of a distorted model is relatively difficult, and the results of the models being distorted cannot be easily transferred to the prototype, as the basic condition of geometric similarity is not satisfied.

PROBLEMS AND EXERCISES

7.1 Define (i) dimensionally homogeneous/rational equation, (ii) prototype, and (iii) physical model or scale model or model, (iii) scale ratio, (iv) undistorted model, and (v) distorted model.

7.2 Describe the Rayleigh's method and the Buckingham's method of dimensional analysis.

7.3 Describe the three kinds of similarities a model should possess.

7.4 Define the following dimensionless numbers and derive expressions for them based on dimensional analysis: (i) Reynolds number, (ii) Froude number, (iii) Weber number, (iv) Euler number, and (v) Mach number.

7.5 Check the dimensional homogeneity of the following equations:

(a) $Q = C_d a\sqrt{2gH}$ (b) $S = ut + 4.905t^2$

7.6 Show that the resistance (R) to the motion of a sphere of diameter (D) falling with a uniform velocity (V) through a real fluid of density (ρ) and viscosity (μ) is given by

$$R = \rho D^2 V^2 f\left(\frac{\mu}{\rho V D}\right)$$

7.7 The discharge (Q) through a horizontal capillary tube depends upon the pressure drop per unit length ($\Delta p/l$), the diameter (D) of the tube and the viscosity (μ) of the fluid. Find the form of the equation.

7.8 A model of an airship was tested in deep water. The length of the model was 10 m and it has a speed of 25 m/s which was measured in water. Determine the speed of the actual-sized ship in air when its length is 250 m. Assume that the kinematic viscosity of air is 13 times that of water. The flows are dynamically similar and the flow is governed by Reynolds law.

7.9 In an open channel, water is flowing at a depth of 2 m. It suddenly forms a jump at a certain point and the depth increases from 2 m to 3.3 m. The velocity of water at 2 m depth is 10 m/s. Another channel was built in which a similar jump was formed. The depth of water in the new channel in which the flow is dynamically similar according to Froude law is 8 m. Calculate the velocity of water in it. Find also the height of the jump in the second case.

7.10 An aeroplane model of scale ratio 1:30 is tested in water, which is 50 times more viscous and 800 times more dense than the air. If the pressure drop in the model during test is 2.3 kg/cm^2, find the pressure drop in the prototype.

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Chapter 8

FLOW THROUGH PIPES

8.1 INTRODUCTION

A pipe is a closed conduit, generally of circular cross-section, used to carry water or any other fluid. In hydraulics, pipes are commonly understood to be conduits of circular cross-section which flow full and the flow is under pressure. City water and gas mains in which flow occurs under pressure are examples of pipes. But if the pipe does not flow full, as is the case of sewer pipes, drainage tiles and culverts, the flow is not under pressure and in such a case the atmospheric pressure exists inside the pipe. The flow is then similar to that of an open channel. In this chapter, we shall consider the flow in the pipes under pressure only.

In Chapter 6, the basic equations of fluid flow were derived with the assumption that the fluid was ideal, i.e. non-viscous. But in nature, there is no fluid which has zero viscosity. The fluids having viscosity are known as practical or real fluids. Due to the presence of viscosity, real fluids differ from non-viscous ideal fluids. By the action of viscosity, the energy supplied to the flowing fluid is converted to thermal energy. The dissipation of energy so caused is a loss of useful energy. Thus, in flow of a real fluid some loss of head takes place.

In this chapter, we consider pipe flow wherein viscous action can be considered to pervade the entire flow. Pipe flow is of great significance in our technology and will always be significant as long as we transport fluid. Also, much valuable experimentation has been performed in pipe flow which has rather general significance. Our first step then is to examine flow in which viscous effects are significant.

8.2 LAMINAR AND TURBULENT FLOWS

Depending on whether the viscosity is dominating or not, the flow of a real fluid is found to be of two types, viz. laminar and turbulent. The limiting conditions which determine whether is flow is laminar or turbulent were first investigated experimentally by Osborne Reynolds in 1883. Reynolds apparatus (Fig. 8.1) consists of a tank containing water and a small tank containing dye. A horizontal glass tube, 1.5 m long and 5 cm in diameter, is fitted to the tank through which water can flow. The flow through the glass tube can be regulated by adjusting the regulating valve. A dye injection arrangement is fitted in the main tank.

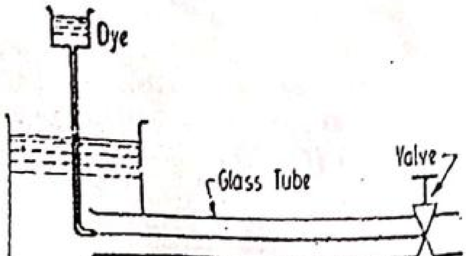
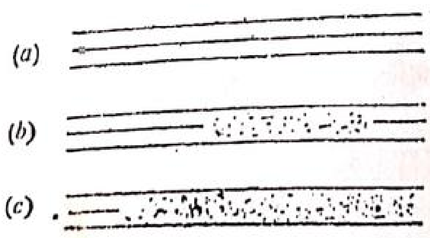


Fig. 8.1 Reynolds experiment



8.2 Dye thread in Reynolds experiment

The water in the tank is allowed to stand for several hours to allow it to come to rest. The outlet valve of the glass tube is then slightly opened. Then a jet of dye, having the same specific weight as that of water, is allowed to enter at the center of the glass tube. It will be seen that a fine thread of dye is carried by the flowing water as shown in Fig. 8.2(a). The dye thread will move so steadily that it will be hardly seen to be in motion. Such a flow is known as laminar or stream line flow.

If we slowly go on increasing the velocity of water through the glass tube, we see that a stage will come when the dye thread will start becoming irregular and then breaking up as in Fig. 8.2(b). The velocity of flow at which the dye thread starts becoming irregular is known as the *lower critical velocity*. If we still go on increasing the velocity of water through the glass tube, we see that the length of the dye thread in the glass tube will start decreasing and ultimately a stage will come when the entire dye thread will disappear (Fig. 8.2c). The velocity at which the whole dye thread is diffused, is known as the *upper critical velocity*. Beyond the upper critical velocity the dye will mix up with water thereby showing violent mixing of water particle in the glass tube. Such a flow is known as turbulent flow.

Reynolds found that the nature of flow in closed conduits depends primarily upon the characteristic dimension of the conduit, the velocity of flow and the density and the viscosity of the fluid. By grouping these variables, Reynolds determined a non-dimensional quantity or parameter denoted by $\rho VL/\mu$. Later on it was known as the Reynolds number after the founder's name.

In case of flow through pipes the characteristic linear dimension L is taken as the diameter of pipe d and the characteristic velocity V is taken as the average velocity. Hence, the Reynolds number is written as

$$Re = \frac{\rho V d}{\mu} = \frac{Vd}{\nu} \qquad (8.1)$$

where ν ($= \mu/\rho$) is the kinematic viscosity.

Reynolds number is very useful in predicting whether the flow is laminar or turbulent and for finding the friction factor f in order to determine the frictional loss of head accurately. Reynolds after carrying out a series of experiments found that flow in a circular pipe is always laminar when $Re < 2000$ and is always turbulent when $Re > 4000$.

The laminar flow occurs when velocity of flow is small and viscous forces are predominant. It is smooth and regular and thus also known as stream line flow. There is practically no influence of fluid particles of one layer over those of the adjacent layers. Diffusion or mixing at molecular level may occur, but macroscopic movement of fluid elements from one layer to another does not occur. Velocity at any point remains nearly constant in magnitude and direction. Such flow rarely occurs in pipes and channels.

When the flow is turbulent the fluid particles no longer move in layers or laminas. Violent mixing of fluid particles takes place due to which they move in chaotic and random manner. As a result, the velocity at any point varies both in magnitude and direction from instant to instant. Flow in pipes and channels are mostly of this type.

Frictional resistance is proportional to the mean velocity of flow when the flow is laminar and to the square of the mean velocity when the flow is turbulent.

It has been found experimentally that when a laminar flow changes into a turbulent flow, it does not change abruptly. But there is a transition between the two types of flow. For usual cases of uniform flow through circular pipes, the flow is assumed to change from laminar to turbulent for Re between 2000 and 4000 and this region is called the *transition zone*. Thus, if $Re < 2000$, flow is laminar and if $Re > 4000$ flow is turbulent in circular pipes in usual conditions of flow and surface roughness. A velocity at which the laminar flow stops, i.e. flow enters from laminar to transition zone is known as the lower critical velocity. A velocity at which turbulent flow starts, i.e. flow enters from transition zone to turbulent is known as the upper critical velocity.

Example 8.1

An oil having kinematic viscosity of 21.4 stokes is flowing through a pipe of 300 mm diameter. Determine the type of flow, if the discharge through the pipe is 15 liters/sec.

Solution

We have, ν = 21.4 stokes = 21.4×10^{-4} m^2/s

Diameter of pipe, d = 300 mm = 0.30 m

$\therefore$ Area of pipe, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.30^2 = 0.0707\ m^2$

Discharge, Q = 15 liters/s = 0.015 m^3/s

$\therefore$ Velocity of flow, $V = \frac{Q}{A} = \frac{0.015}{0.0707} = 0.212\ m/s$

The Reynolds number of the flow is given by

$$Re = \frac{Vd}{\nu} = \frac{0.212 \times 0.30}{21.4 \times 10^{-4}} = 29.75$$

As the Reynolds number is less than 2000, the flow is laminar.

8.3 VELOCITY DISTRIBUTION IN LAMINAR AND TURBULENT FLOWS

Velocity distribution curves for a circular pipe are shown in Fig. 8.3. When the flow is laminar, the velocity varies along any diameter as shown by curve A. The velocity is zero at the pipe walls and increases gradually until the maximum velocity is reached at the pipe center. In this case the velocity varies as the ordinates of a paraboloid of revolution and the maximum velocity V_c at the center of the pipe is twice the average velocity.

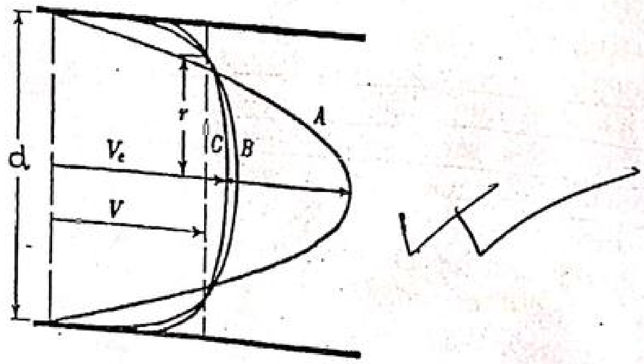


Fig. 8.3 Velocity distribution in straight pipe

When the flow is turbulent, the velocity distribution curves are more flatter, as indicated in Fig. 8.3. The velocity is practically zero at the pipe walls, but increases more rapidly for a short distance from the walls than in laminar flow. Throughout the central zone, however, the mixing resulting from turbulence tends to equalize the velocities of the particles. Turbulence increases with Reynolds number. So, the velocity distribution becomes more uniform as the Reynolds number increases. Tests have shown that the ratio of the average to maximum velocity (V/V_c) in a pipe of circular cross-section varies with the Reynolds number Re approximately as shown in the following table.

Re	V/V_c
≤ 1700	0.50
2000	0.55
3000	0.71
5000	0.76
10000	0.78
30000	0.80
≥ 100000	0.81

8.4 LOSS OF HEAD

Loss of head in m of fluid, meaning loss of energy expressed in m-kg/kg or m-N/N, occurs in any flow of fluid through a pipe. The loss is caused by: (1) pipe friction along the straight sections of pipe of uniform diameter and uniform roughness, and (2) changes in velocity or direction of flow. Losses of these two types are ordinarily referred to respectively as major losses and minor losses.

Major Loss: This is a continuous loss of head, h_f, assumed to occur at a uniform rate along the pipe as long as the size and quality of pipe remain constant, and is commonly referred to as the loss of head due to pipe friction.

Minor Losses: These consist of

1. A loss of head, h_c, due to contraction of cross-section. This loss is caused by a reduction in the cross-sectional area of the stream and the resulting increase in velocity. The contraction may be sudden or it may be gradual. The loss of head at the entrance to a pipe from a reservoir is a special case of loss due to contraction.
2. A loss of head, h_e, due to enlargement of cross-section. This loss is caused by an increase in the cross-sectional area of the stream with resulting decrease in velocity. The enlargement may be either sudden or gradual. The loss of head at the outlet end of a pipe where it discharges into a reservoir is a special case of loss of head due to enlargement.
3. A loss of head, h_g, caused by obstructions such as gates or valves which produces a change in cross-sectional area in the pipe or in the direction of flow. The result is usually a sudden increase or decrease in velocity followed by a more gradual return to the original velocity.
4. A loss of head, h_b, caused by bends or curves in pipes, in addition to the loss which occurs in an equal length of straight pipe. Such bends may be of any total deflection angle as well as any radius of curvature. Occasionally, as in a reducing elbow, the loss due to the bend is superimposed on a loss due to change in velocity.

If the symbol H is used to designate all losses of head in a pipe line in which there is steady continuous flow, then

$$H = h_f + h_c + h_e + h_g + h_b \tag{8.2}$$

In a long pipe the major loss of head is due to friction in the pipe only. The minor losses are small compared to the friction and can be neglected altogether. But in the case of a short pipe, the minor losses, as compared to the friction loss, are of appreciable magnitude and must be included.

Strictly speaking, there is no hard and fast rule to define a long pipe. But a pipe is generally termed as a *long pipe* when its length is more than 1000 times its diameter, and it is termed as a *short pipe* if its length is less than 1000 times its diameter.

8.5 FRICTIONAL LOSS WITH LAMINAR FLOW: HAZEN-POISEUILLE EQUATION

Figure 8.4 represents a longitudinal section and a cross-section of a straight horizontal circular pipe of constant diameter d in which a fluid of uniform specific weight γ is moving from left to right with steady laminar motion. Consider a circular cylinder of fluid, abcd, of length L extending from section 1 where the pressure is $p_1 = \gamma h_1$ to section 2 where the pressure has decreased to $p_2 = \gamma h_2$. The difference in total pressure force on the two ends of the cylinder is thus

$$\gamma(h_1 - h_2)\pi y^2 = \gamma h_f \cdot \pi y^2 \tag{8.3}$$

It is considered that the cylinder is in equilibrium between this pressure difference and the shear resistance exerted by the surrounding fluid on the curved surface of the cylinder.

From the definition of viscosity, the unit shear stress on this surface is

$$\tau = \mu\left(\frac{-dv}{dy}\right)$$

since for each increment dy in distance from the pipe axis, there is a decrease dv in velocity. The total shear stress on the surface of the cylinder is thus

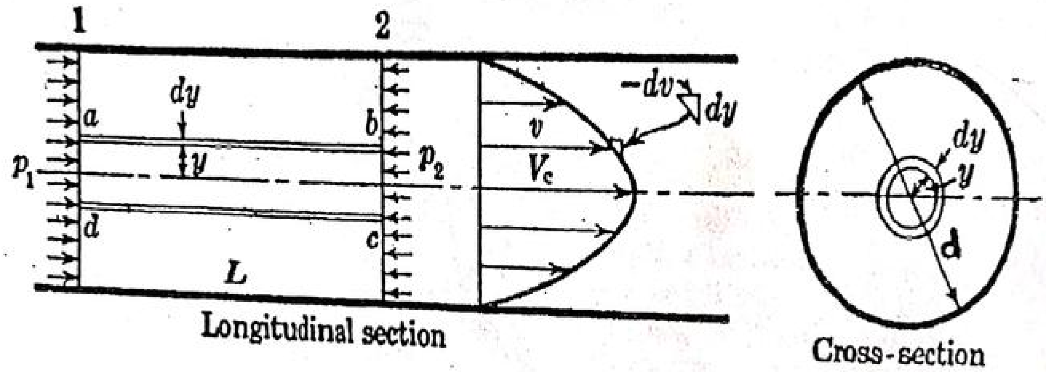


Fig. 8.4 Laminar flow in a pipe

$$-2\pi y L\mu\frac{dv}{dy} \tag{8.4}$$

Equating Eqs.(8.3) and (8.4) leads to a simple differential equation

$$dv = -\frac{\gamma h_f}{2L\mu}ydy \tag{8.5}$$

Integrating

$$v = -\frac{\gamma h_f y^2}{4L\mu} + C_1 \tag{8.6}$$

When y = d/2, v = 0. Therefore

$$C_1 = \frac{\gamma h_f d^2}{16L\mu}$$

Substituting this value in Eq. (8.6), we get

$$v = \frac{\gamma h_f}{4L\mu}\left(\frac{d^2}{4} - y^2\right) \tag{8.7}$$

This equation gives the velocity v at any distance y from the pipe axis. The discharge through the ring of width dy is

$$dQ = v \times 2\pi y dy$$

Substituting the value of v from Eq, (8.7)

$$dQ = 2\pi\frac{\gamma h_f}{4L\mu}\left(\frac{d^2}{4}y - y^3\right)dy \tag{8.8}$$

Integrating between the limits y = 0 and y = d/2, we obtain

$$Q = \frac{\pi\gamma h_f d^4}{128L\mu} \tag{8.9}$$

or the loss of head, substituting ρg for γ and ν for μ/ρ

$$h_f = \frac{128L\mu Q}{\pi\gamma d^4} = \frac{128L\mu Q}{\pi\rho g d^4} = \frac{128L\nu Q}{\pi g d^4} \quad (8.10)$$

Since

$$Q = \frac{\pi}{4}d^2V$$

where V is the cross-sectional mean velocity, we get

$$h_f = \frac{32L\nu V}{gd^2} \quad (8.11)$$

Equation (8.11) was first determined experimentally by Hagen in 1839 and simultaneously by Poiseuille in 1840 and it is usually known as the Hazen-Poiseuille equation. It indicates that in laminar flow the loss of head is proportional to the first power of the mean velocity. It was experimentally determined that Eq. (8.11) gives correct result. It is mostly used for the experimental determination of fluid viscosity by measuring the loss of head in a pipe of length L. Equation (8.11) can be written for the fluid viscosity as

$$\mu = \frac{\gamma h_f d^2}{32LV} = \frac{\gamma(h_1 - h_2)d^2}{32LV} = \frac{(p_1 - p_2)d^2}{32LV} \quad (8.12)$$

Equation (8.7) shows that the velocity distribution along the diameter is parabolic, the maximum velocity being at the center of the pipe (y = 0) and having the value

$$V_c = \frac{h_f g d^2}{16L\nu} \quad (8.13)$$

Also, rearranging Eq. (8.11)

$$V = \frac{h_f g d^2}{32L\nu} \quad (8.14)$$

From the last two equations, we get

$$V_c = 2V \quad (8.15)$$

Example 8.2

In a laboratory experiment, a crude oil is flowing through a pipe of 50 mm diameter with a velocity of 1.5 m/s. During this experiment, a pressure difference of 180 N/mm^2 was recorded from two pressure gauges 8 m apart. Find the viscosity of the flowing oil.

Solution Given, diameter of pipe, d = 50 mm = 0.05 m

Velocity of oil, V = 1.5 m/s

Pressure difference, $p_1 - p_2 = \gamma h_f$ = 180 N/mm^2 = 180×10^{-6} N/m^2

Length of the pipe, L = 8 m

$$\therefore \mu = \frac{\gamma h_f d^2}{32LV} = \frac{180 \times 10^{-6} \times 0.05^2}{32 \times 8 \times 1.5} = 11.72 \times 10^{-10} \text{ N-s/m}$$

8.6 FRICTIONAL LOSS WITH TURBULENT FLOW: DARCY-WEISBACH FORMULA

The following discussion applies to all liquids and approximately to gases when the pressure drop is not more than 10 percent of the initial pressure. Consider a straight pipe of internal diameter d in which a fluid is flowing at a mean velocity V. Let the loss of head in length L be denoted by h_f.

Certain general laws based upon observation and experiment appear to govern fluid friction in pipes and are expressed in all the generally accepted pipe formulas. These laws briefly stated are:

1. Frictional loss in turbulent flow generally increases with the roughness k of the pipe. When the flow is laminar the frictional loss is independent of the roughness (Art. 8.5).

2. Frictional loss is directly proportional to the area of the wetted surface, or to πdL.

3. Frictional loss varies inversely as some power of the pipe diameter, or as $1/d^x$.

4. Frictional loss varies as some power of the velocity, or as V^n.

5. Frictional loss varies as some power of the ratio of viscosity to density of the fluid, or as $(\mu/\rho)^r$.

Combining these factors, a rational equation for loss of head due to pipe friction for any fluid can be written in the form

$$h_f = K' \times k \times \pi dL \times \frac{1}{d^x} \times V^n \times \left(\frac{\mu}{\rho}\right)^r$$

$$= \left[K'k\pi\left(\frac{\mu}{\rho}\right)^r\right] \times \frac{L}{d^m} \times V^n \qquad (8.16)$$

where K' is a constant of proportionality and $m = x - 1$.

The effect of viscosity and density of water on loss of head at usual flow velocities is so small that it can be neglected. If there is any little effect, it could be easily included in a general coefficient. So, if we substitute K for the quantity in brackets in Eq.(8.16), the base formula for loss of head in pipe flow can thus be stated as

$$h_f = K\frac{L}{d^m}V^n \qquad (8.17)$$

A determination of K, m and n is necessary for practical application of Eq. (8.17) to flow problems. Chezy in 1775 pointed out that the loss of head in the flow of water in conduits varied approximately as the square of the mean velocity. About the middle of the nineteenth century, Darcy, Weisbach and others, accepted Chezy's value of 2 for n, further modified Eq.(8.17) by proposing a value of 1 for m, and divided and multiplied Eq.(8.17) by 2g, so that

$$h_f = (K \times 2g) \times \frac{L}{d} \times \frac{V^2}{2g} \qquad (8.18)$$

By substituting the friction factor f for $(K \times 2g)$, we obtain

$$h_f = f\frac{L}{d}\frac{V^2}{2g} \qquad (8.19)$$

Equation (8.19) is the well-known pipe formula, known as the Darcy-Weisbach formula. It is in a very convenient form since it expresses the loss of head in terms of the velocity head in the pipe. Moreover, it is dimensionally correct since f is dimensionless, L/d is a ratio and h_f and $V^2/2g$ are both expressed in units of length.

Equation (8.19) is frequently written as

$$h_f = 4f'\frac{L}{d}\frac{V^2}{2g} \qquad (8.20)$$

This equation is known as the Fanning equation. The Fanning friction factor f' is one-fourth of the Darcy-Weisbach friction factor f.

The limitations of the Darcy-Weisbach formula are:

1. The loss of head with turbulent flow varies not only as the square of the mean velocity, but as some power varying from 1.7 to 2 or more depending on the roughness of the pipe. This discrepancy must be taken care of by varying the value of f. For laminar flow, the loss of head varies as the first power of the mean velocity (Art. 8.5).

2. Since $V = Q/A = Q/(\pi d^2/4)$, for a given Q, f and L, the loss of head by the Darcy-Weisbach formula varies inversely as the fifth power of the diameter. Tests have shown that the actual variation is closer to the 5.25 power and that the exponent of d in the formula

should be close to 1.25. Again the discrepancy is taken care of by varying the value of f. In laminar flow the loss of head varies inversely as the fourth power of the diameter.

3. The friction factor must therefore be a function of velocity and diameter as well as of the pipe roughness and of the viscosity and density of the fluid.

Remarks

1. For a given velocity of flow, the value of f decreases as the diameter of pipe increases. This decrease of f is due to the decrease in relative roughness k/d of the material in the pipe wall.

2. Some kinds of pipe become rougher with age with resulting increase in f. This possibility is usually taken care in design by increasing the value of f for new pipe by a certain percentage. The increase in f for cast iron or steel pipe may be 50% to 100% after some years of service. However, wood pipe and asbestos cement pipe have shown little or no increase in f after many years of service.

Example 8.3

Find the loss of head due to friction in a pipe of 1 m diameter and 15 km long. The velocity of water in the pipe is 1 m/s. Take f = 0.020 and neglect minor losses. Use the Darcy-Weisback formula.

Solution We have, d = 1 m, L = 15 km = 15000 m, V= 1 m/s, f = 0.020

$$\therefore h_f = f\frac{L}{d}\frac{V^2}{2g} = 0.020 \times \frac{15000}{1} \times \frac{1^2}{2 \times 9.81} = 15.29\ m \text{ of water}$$

8.7 SMOOTH AND ROUGH PIPES

Nikuradse Equivalent Sand Grain Roughness (k)

Nikuradse in 1933 conducted a series of experiments on flow through pipes which were artificially roughened by gluing sand grains of uniform diameter. He introduced the concept of equivalent sand grain roughness (k) as standard for all other types of roughness elements. The ratio k/d of the equivalent sand grain roughness to the pipe diameter is known as the *relative roughness*.

Table 8.1 gives the values of equivalent sand grain roughness for different pipe materials.

Table 8.1 Equivalent sand grain roughness (k) for various pipe materials

Sl. No.	Pipe material	k(mm)
1.	Glass	0.0003
2.	Wrought iron, steel	0.046
3.	Asphalted cast iron	0.12
4.	Galvanized iron	0.15
5.	Cast iron	0.26
6.	Concrete	0.30 – 3.0
7.	Riveted steel	0.90 – 9.0

The above values correspond to the material in new and clean condition. As the pipe becomes older, the roughness increases due to erosion.

Laminar Sublayer

Even in turbulent flow there exists next to the wall of the pipe a very thin layer in which the flow is laminar. This layer is known as the laminar or viscous sublayer. The thickness of this layer is given by

$$\partial_v = \frac{11.6\nu}{u^*} \qquad (8.21)$$

where $u^* = \sqrt{\tau/\rho}$ is the shear or friction velocity and τ is the shear stress on the pipe wall.

Hydraulically Smooth and Rough Boundaries

A pipe is said to be hydraulically smooth if the height of the roughness elements is less than the thickness of the laminar sublayer ($k < \partial_v$), i.e. the roughness elements are well-covered by the laminar sublayer. In such a case, variations in relative roughness k/d do not affect the value of the friction factor f. On the other hand, if the height of the roughness elements are greater than the thickness of the laminar sublayer ($k > \partial_v$), their presence affects the amount of turbulence and hence the value of f and the pipe is said to be hydraulically rough. As the height of the roughness elements k increases or the thickness of the laminar sublayer decreases with increasing Reynolds number, the turbulence increases to a maximum level at which it is said to be "fully developed".

8.8 DETERMINATION OF FRICTION FACTOR f

Laminar Flow

When the flow is laminar, the Hazen-Poiseuille equation, Eq. (8.11), applies. Equation (8.11) can be put in the Darcy-Weisbach form by multiplying numerator and denominator by 2V and replacing Vd/ν by the Reynolds number Re. Thus,

$$h_f = \frac{64}{\mathrm{Re}}\frac{L}{d}\frac{V^2}{2g} \qquad (8.22)$$

from which it is apparent that, for laminar flow

$$f = \frac{64}{\mathrm{Re}} \qquad (8.23)$$

Equation (8.23) indicates that when the flow is laminar, the loss of head depends on Re, but is independent of the roughness of the pipe.

Turbulent Flow

When the flow is turbulent, the value of f depends not only on the Reynolds number, but also on the relative roughness of the pipe. Blasius was the first to give an empirical equation for finding the friction factor f, as early as 1913, based on his experimental work on hydraulically smooth pipe in turbulent flow. The formula developed by Blasius that is valid up to about $\mathrm{Re} = 10^5$ is given by

$$f = \frac{0.3164}{\mathrm{Re}^{1/4}} \qquad (8.24)$$

Studies by Prandtl and von Karman led to the following equations for determining f for the two extreme conditions of flow in pipes.

For smooth pipes

$$\frac{1}{\sqrt{f}} = 2\log\left(\frac{\mathrm{Re}\sqrt{f}}{2.51}\right) \qquad (8.25)$$

For rough pipes

$$\frac{1}{\sqrt{f}} = 2\log\left(3.7\frac{d}{k}\right) \quad (8.26)$$

These equations are regarded as great scientific achievement as they have been verified by experiment for pipes of all sizes and for different liquids. They show that for turbulent flow in pipes which are hydraulically smooth, f is independent of the relative roughness and is a function of the Reynolds number only. On the other hand, for turbulent flow in pipes which are hydraulically rough and turbulence is fully developed, f is independent of the Reynolds number and depends only on the relative roughness.

Between these two limiting conditions of flow, there is a transition region for which White and Colebrook developed the following formula for f for use with commercial pipes:

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{k}{3.7d} + \frac{2.51}{Re\sqrt{f}}\right) \quad (8.27)$$

→ For any pipe

The transition region merges at one end into the smooth pipe flow and at other end into the zone of fully developed turbulence.

Example 8.4

A discharge of 900 liters per minute of water takes place in a pipe 15 cm in diameter and having a roughness height of 0.07 cm. Determine the friction factor f by the White-Colebrook formula. Take ν = 1 centipoise.

Solution We have, Q = 900 liters/min = 0.015 m^3/s, d = 15 cm = 0.15 m, k = 0.07 cm = 0.0007 m, $V = Q/A = 0.015/(\pi \times 0.15^2/4) = 0.85$ m/s, $\nu = 1\times10^{-2}\times10^{-4} = 10^{-6}$ m^2/s, $Re = Vd/\nu = 0.85\times0.15/10^{-6} = 127500$

Now, the White-Colebrook formula is

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{k}{3.7d} + \frac{2.51}{Re\sqrt{f}}\right)$$

or

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{0.0007}{3.7\times0.15} + \frac{2.51}{127500\sqrt{f}}\right)$$

or

$$\frac{1}{\sqrt{f}} = -2\log\left(0.00126 + \frac{1.96\times10^{-5}}{\sqrt{f}}\right)$$

The value of f is determined by trial. Assume values of f and compute L.H.S. and R.H.S. The value of f for which L.H.S. = R.H.S. is the required value of f.

Assumed f	L.H.S.	R.H.S.
0.01	10	5.673
0.02	7.071	5.708
0.03	5.774	5.724
0.031	5.678	5.725

∴ The friction factor, f = 0.031.

8.9 STANTON AND MOODY DIAGRAMS

As already stated, Nikuradse conducted a series of experiments on smooth and artificially roughened pipes by gluing sand on the interior pipe surface. The results of Nikuradse are generally plotted showing the variation of f against the Reynolds number on logarithmic scale (Fig. 8.5). The plot of friction factor against the Reynolds number on a log-log chart is called a Stanton diagram.

From Fig. 8.5, it is obvious that Nikuradse's data cover both laminar and turbulent ranges. For Re < 2300, there is a simple relation between f and Re, given by Eq.(8.23) and completely independent of roughness. On log-log paper, the f–Re relationship is a straight line.

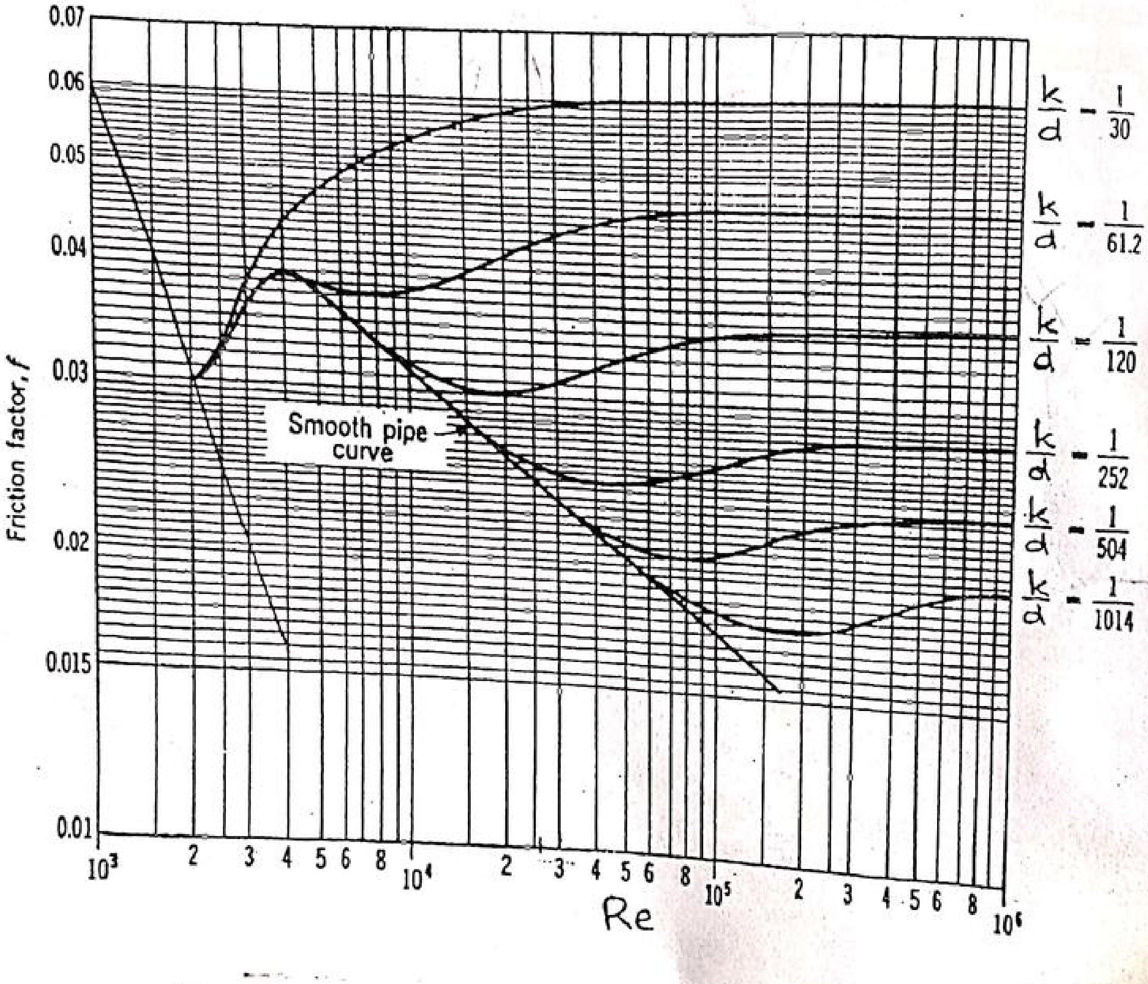


Fig. 8.5 Nikuradse's data for artificially roughened pipe flows

In the turbulent part of the plot, after the transition region, all roughness curves are coincident with the smooth-pipe curve. Later, each curve departs from the smooth-pipe curve in a sequence such that the greater the roughness, the earlier the departure. That part of any curve coincident with the smooth-pipe curve is called the *smooth-pipe zone of flow*. Note that after passing the smooth-pipe zone, each curve eventually flattens to a straight line parallel to the abscissa. Here the friction factor f is independent of the Reynolds number. This region for each curve is called the *rough-pipe zone of flow*. Thus, each curve except that of the smooth pipe goes through three zones of flow, the position and extent of each zone depending on the roughness of the pipe.

Nikuradse data have been developed for artificial conditions of roughness. There is the question of how well this type of roughness approximates actual conditions of roughness as found in real situations. Moody in 1944 has constructed one of the most convenient charts for determining friction factors in clean, commercial pipes. This chart is a Stanton diagram that expresses f as a function of relative roughness and the Reynolds number and is shown in Fig. 8.6.

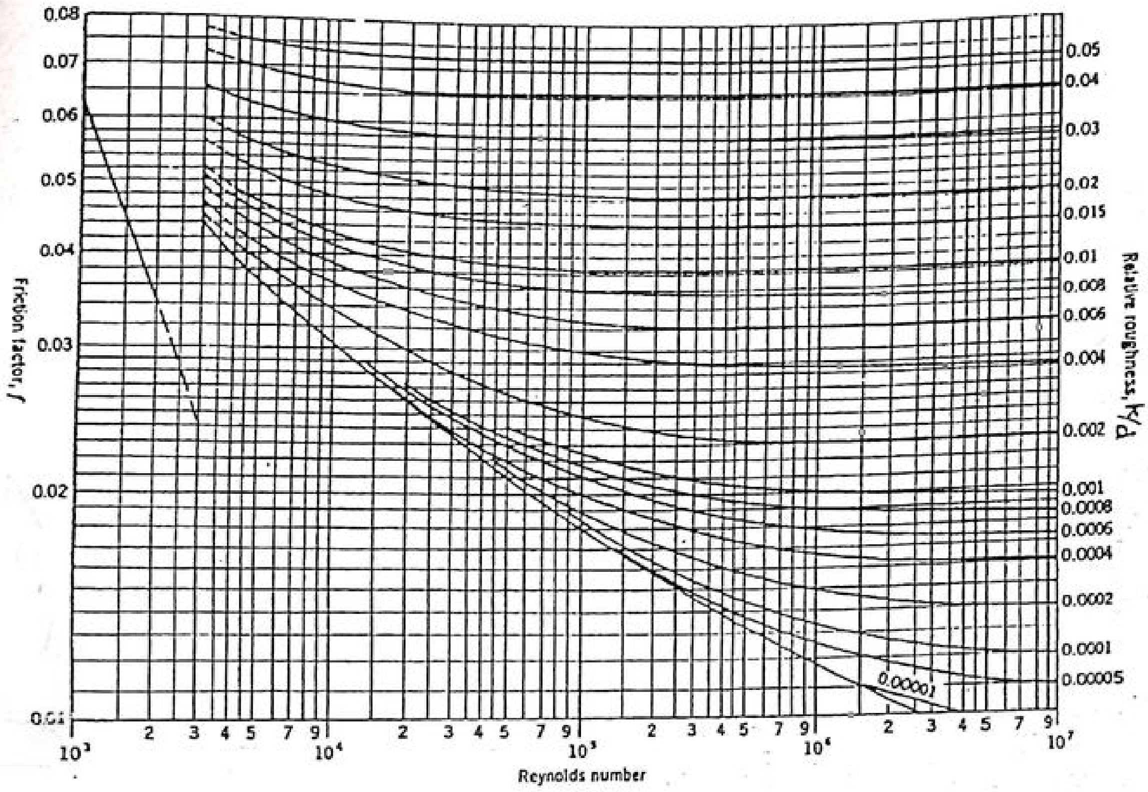


Fig. 8.6 Moody diagram

8.10 OTHER PIPE FORMULAS

Chezy Formula

Using $h_f/L = S$ and $d = 4R$, where S is the slope of the energy grade line (Fig. 8.7) representing the loss of head per m of pipe and R (=A/P) is the hydraulic radius, the Darcy-Weisbach formula can be put in the form

$$V = \sqrt{\frac{2g}{f}} \times \sqrt{4R} \times \sqrt{S} = \sqrt{\frac{8g}{f}}\sqrt{RS} \quad (8.28)$$

Substituting a coefficient C for $\sqrt{8g/f}$, we obtain

$$V = C\sqrt{RS} \quad (8.29)$$

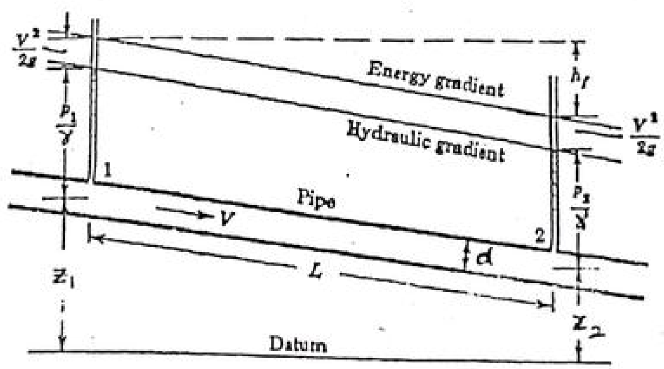


Fig. 8.7 Frictional loss in pipe

This formula for velocity of flow in terms of the hydraulic radius of a conduit and slope of the energy grade line is called the Chezy formula. The Chezy coefficient C is a function of the same variables as the Darcy-Weisbach friction factor f and the Chezy formula is therefore subject to same defects as noted in Art. 8.6 for the Darcy-Weisbach formula.

Manning Formula

The Manning formula is one of the best-known open channel formulas and is commonly used for pipes. In the form of Eq.(8.29), the Manning formula is

$$V = \frac{1}{n} R^{2/3} S^{1/2} \tag{8.30}$$

where n is the roughness coefficient.

Hazen-Williams Formula

This well-known formula has been mostly used by water works and irrigation engineers. This formula is

$$V = 0.25C_1 R^{0.63} S^{0.54} \tag{8.31}$$

This formula was designed for the flow of water in both pipes and open channels, but is used mostly for pipes.

Example 8.5

Water flows through a pipe 200 mm in diameter and 60 m long with a velocity of 2.5 m/s. Find the head lost in friction by using the Chezy formula, assuming C = 55 $m^{1/2}/s$.

Solution We have, d = 200 mm = 0.20 m, L = 60 m, V = 2.5 m/s, C = 55 $m^{1/2}/s$

$$R = \frac{d}{4} = \frac{0.20}{4} = 0.05\ m$$

From the Chezy formula $V = C\sqrt{RS}$, we have

$$2.5 = 55 \times \sqrt{0.05 \times S}$$

$$\therefore\ S = \frac{2.5^2}{55^2 \times 0.05} = 0.04113$$

$\therefore$ The head loss, $h_f = S \times L = 0.04113 \times 60 = 2.48$ m of water

8.11 MINOR LOSSES

Loss of Head due to Contraction: It is given by

$$h_c = K_c \frac{V^2}{2g} \tag{8.32}$$

where K_c is an empirical coefficient and V is the velocity in the smaller pipe (Fig. 8.8). The value of K_c depends on V and the ratio of the smaller to the larger diameter. This loss is usually taken to be equal to

$$h_c = 0.5 \frac{V^2}{2g} \tag{8.33}$$

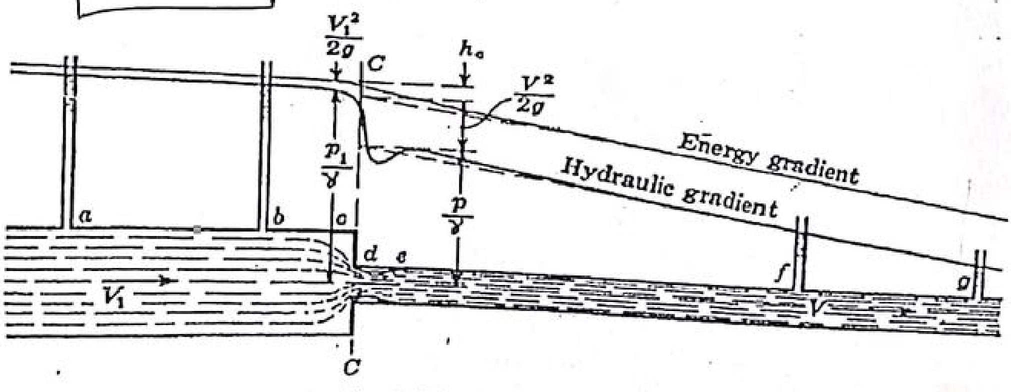


Fig. 8.8 Sudden contraction in pipe

The loss of head at the entrance to a pipe from a reservoir is also taken to be the same.

Loss of Head due to Enlargement: It is usually given by

$$h_e = \frac{(V - V_1)^2}{2g} \tag{8.34}$$

where V is the velocity in the smaller pipe, V_1 is the velocity in the larger pipe (Fig. 8.9). Equation (8.34) is known as the Carnot or Borda equation. The loss of head at the outlet to a pipe which discharges into a reservoir or into the atmosphere is usually taken to be equal to

$$h = \frac{V^2}{2g} \tag{8.35}$$

where V is the velocity in the pipe.

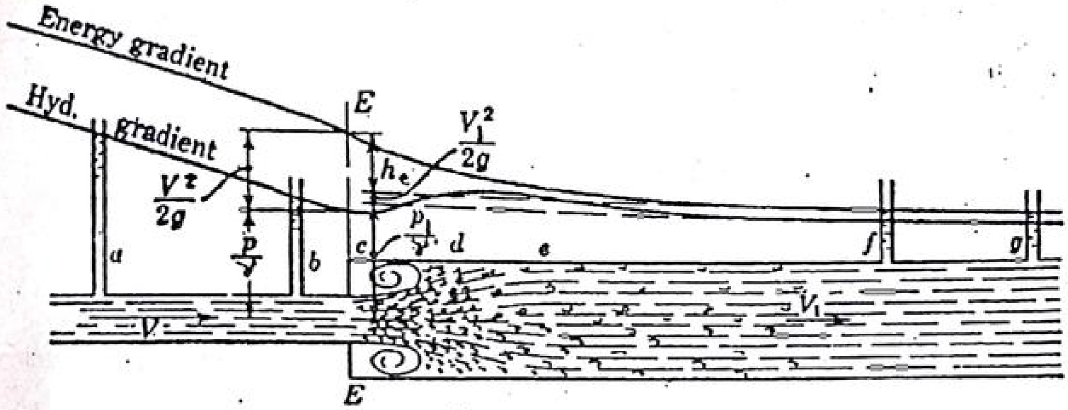


Fig. 8.9 Sudden enlargement in pipe

Loss of Head due to Obstructions: The loss of head in pipes due to gates, valves and other obstructions is given by

$$h_g = K_g \frac{V^2}{2g} \tag{8.36}$$

where V is the mean velocity in the pipe. The coefficient K_g depends on the size of obstruction compared to the diameter of the pipe.

Loss of Head due to Bends: This loss of head is usually expressed as a function of the velocity head in the pipe, i.e.

$$h_b = K_b \frac{V^2}{2g} \tag{8.37}$$

The value of the coefficient K_b varies with the ratio of the radius of curvature of the bend r to the pipe diameter d, the roughness of the surface in the bend and the Reynolds number. If the flow is turbulent, the effect of variation in Reynolds number is not of much practical importance and in that case K_b is a function of r/d and the roughness of the bend.

Example 8.6

A pipe 60 m long and 15 cm in diameter is connected to a water tank at one end and flows freely into the atmosphere at the other end. The height of the water level in the tank is 2.6 m above the center of the pipe. The pipe is horizontal and f = 0.04. Determine the discharge through the pipe in liters/sec, if all the minor losses are to be considered.

Solution We have, L = 60 m, d = 15 cm = 0.15 m, H = 2.6 m, f = 0.04

Since the pipe is connected to a water tank and discharges freely into the atmosphere, there are contraction loss at the entrance and a sudden enlargement loss at the outlet. The contraction loss is given by $h_c = 0.5V^2/2g$ and the sudden enlargement loss is given by $h_e = V^2/2g$, where V is the velocity of water through the pipe.

$$\therefore H = f\frac{L}{d}\frac{V^2}{2g} + 0.5\frac{V^2}{2g} + \frac{V^2}{2g} = \frac{V^2}{2g}(f\frac{L}{d} + 1.5)$$

or

$$2.6 = \frac{V^2}{2\times 9.81}(0.04\times\frac{60}{0.15} + 1.5)$$

$$\therefore V^2 = \frac{2.6\times 2\times 9.81}{17.5} = 2.915$$

$$\therefore V = 1.7073\ m/s$$

$$\therefore Q = AV = \frac{\pi}{4}\times 0.15^2\times 1.7073 = 0.0302\ m^3/s = 30.2 \text{ liters/sec}$$

PROBLEMS AND EXERCISES

8.1 What do you mean by lower critical velocity and upper critical velocity?

8.2 How can you physically identify whether the flow in a pipe or channel is laminar or turbulent?

8.3 Draw the velocity distribution curves at the cross-section of a pipe for laminar and turbulent flows and state the type of velocity variation.

8.4 Why does the loss of head occur in pipe flow? Describe the different types of losses that occur in pipe flow.

8.5 Derive the Hazen-Poiseuille equation for laminar flow in a pipe. Using this equation, show that the maximum velocity is twice the mean velocity.

8.6 Derive the Darcy-Weisbach formula for turbulent flow in a pipe. What are the limitations of this formula? State the SI unit(s) of the friction factor f.

8.7 What do you mean by Stanton and Moody diagrams?

8.8 Write short notes on (i) laminar flow, (ii) turbulent flow, (iii) laminar or viscous sublayer, (iv) hydraulically smooth boundary, and (v) hydraulically rough boundary.

8.9 An oil of specific gravity 0.85 is flowing through a pipe of 5 cm diameter at the rate of 3 liters/sec. Find the type of flow, if the viscosity of the oil is 3.8 poises.

8.10 A pipeline 30 in diameter and 3200 m long is used to pump 50 kg per second of an oil whose specific weight is 950 kg/m^3 and whose kinematic viscosity is 2.1 stokes. Find the head loss.

8.11 Show that the energy coefficient α and the momentum coefficient β for laminar flow in pipe are equal to 2 and 4/3, respectively.

8.12 The difference of heads between two ends of a pipe, 250 m long and 300 mm in diameter, is 1.5 m. Taking f = 0.04 and neglecting minor losses, calculate the discharge flowing through the pipe. Use the Darcy-Weisbach formula.

8.13 A concrete pipe 20 cm in diameter and having k = 0.06 cm carries water at the rate of 0.02 m^3/s. Determine the friction factor f by the White-Colebrook formula. Take $v = 10^{-6}$ m^2/s.

8.14 Determine the friction factor f by the White-Colebrook formula when a riveted steel pipe 10 cm in diameter carries water at a velocity of 6 m/s, k = 0.3 mm and $v = 10^{-6}$ m^2/s.

8.15 A town having a population of 100000 is to be supplied with water from a reservoir 5 km apart and it is stipulated that one half of the daily supply of 150 liters per head should be delivered in 8 hours. What must be the size of the pipe to furnish the supply, if the head available is 12 m? Take $C = 45\ m^{1/2}/s$ in the Chezy formula.

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PIPE FLOW PROBLEMS

9.1 INTRODUCTION

A pipe line is made of small lengths of pipes of uniform or different diameters, joined together with the help of pipe fittings such as couplings, flanges, reducers, bends, elbows and pipe junctions. Pipe fittings like valves and sluices are employed for the regulation of flow. Long pipes are used to convey different kinds of fluids such as water, oil, gas, etc. For water supply installations, pipe line is meant to distribute water from one reservoir to one or more reservoirs.

9.2 PIPES IN SERIES

Figure 9.1 represents a system of pipes of different lengths and diameters conveying liquid from one reservoir to another. Such a pipe line is called a compound pipe or pipes in series. As the pipes are in series, the discharge through them will be continuous. Assuming the liquid at rest in both reservoirs, the difference H in elevation of free surfaces is the total head producing discharge. The losses of head, as indicated by the drops in the energy gradient are successively: h_{c1}, due to contraction at entrance at A; h_{f1}, due to friction in pipe 1; h_{c2}, due to contraction to smaller pipe at B; h_{f2}, due to friction in pipe 2; h_{e1}, due to enlargement to larger pipe at C; h_{f3}, due to friction in pipe 3; and h_{e2}, due to enlargement at outlet at D. Therefore,

hc1 + hf1 + hc2 + hf2 + he1 + hf3 + he2

$$H = h_{c1} + h_{f1} + h_{c2} + h_{f2} + h_{e1} + h_{f3} + h_{e2} \qquad (9.1)$$

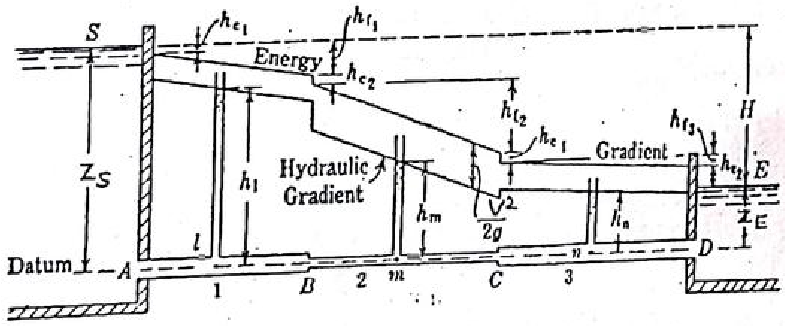

Fig. 9.1 System of pipes connecting two reservoirs

The hydraulic gradient is at a distance $V^2/2g$ below the energy gradient at all points in the three pipes. Since pipe 2 is smaller than pipe 1, the velocity head is greater and the hydraulic gradient is farther below the energy gradient. With the enlargement at C, however, the velocity head becomes less, resulting in a rise in the hydraulic gradient at that point.

In most hydraulic problems the major pipe friction losses h_{f1}, h_{f2} and h_{f3} in Eq.(9.1) constitute most of the total head H, and the minor losses are mostly so small as to be negligible. If the pipe length is about 500 pipe diameters, the error resulting from neglecting minor losses will ordinarily not exceed 5 per cent, and if the pipe length is 1000 diameters or more, the effect of minor losses can usually be considered negligible. However, if it is desired to include these losses, a solution may be obtained first neglecting them and then correcting the results by including them.

Figure 9.2 shows a simplified diagram of flow through a pipe line of different diameters in series connecting two reservoirs. Minor losses are negligible and only the hydraulic gradient is shown. The flow is assumed to be continuous and steady. Two common problems which arise are:

1. Sizes and lengths of pipe and Q given, to find the total loss of head. This problem can be solved as follows. The total loss of head is given by

$$H = h_{f1} + h_{f2} + h_{f3}$$

$$= f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} + f_2 \frac{L_2}{d_2}\frac{V_2^2}{2g} + f_3 \frac{L_3}{d_3}\frac{V_3^2}{2g} \qquad (9.2)$$

Since $f\dfrac{L}{d}\dfrac{V^2}{2g} = f\dfrac{L}{d}\dfrac{Q^2}{2gA^2} = f\dfrac{L}{d}\dfrac{Q^2}{2g}\dfrac{1}{(\pi d^2/4)^2} = \dfrac{8Q^2}{\pi^2 g}\dfrac{fL}{d^5}$, therefore

$$H = \frac{8Q^2}{\pi^2 g}\left(\frac{f_1 L_1}{d_1^5} + \frac{f_2 L_2}{d_2^5} + \frac{f_3 L_3}{d_3^5}\right) \qquad (9.3)$$

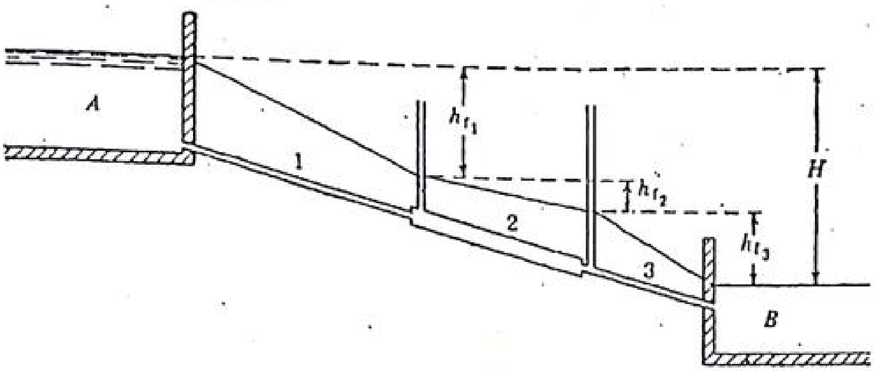

Fig. 9.2 Pipe lines of different diameters in series

Now, if the coefficient of friction is the same for all the pipes, then Eqs.(9.2) and (9.3) become

$$H = \frac{f}{2g}\left(\frac{L_1 V_1^2}{d_1} + \frac{L_2 V_2^2}{d_2} + \frac{L_3 V_3^2}{d_3}\right) \qquad (9.4)$$

and

$$H = \frac{8Q^2 f}{\pi^2 g}\left(\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right) \qquad (9.5)$$

respectively. The minor losses can be computed and included if considerable.

2. Allowable losses of head and lengths and sizes of pipes given, to find Q. Four different methods of solution of this problem using Eq. (9.4) or (9.5) are as follows.

i) Trial solution: Assume a Q. Compute loss of head in each pipe by formula or diagram and add the losses. Compare with the allowable loss and revise Q in proper direction. Repeat until satisfactory check is obtained.

ii) Algebraic solution: Write $f\dfrac{L}{d}\dfrac{V^2}{2g}$ for each pipe, assuming values of f and equating sum of the terms to the allowable head loss. Express all velocity heads in terms of velocity head in one of the given sizes of pipes. Solve for the velocity head and velocity. Compute the velocities in the other pipes. Look up proper f's and check total loss of head. Revise solution if necessary.

This method is adapted to the condition in which minor losses are appreciable since they can also be expressed in terms of the velocity head and included in the equation. The minor loss coefficients selected for the first solution may also need revision.

iii) Equivalent-length solution: Reduce the overall length of compound pipe to an equivalent length of some selected diameter. With this selected diameter and computed equivalent length, determine Q for the given loss of head. Using this Q, check summation of losses in the line. A pipe diagram is of much assistance in the application of this method.

iv) Equivalent-diameter solution: Reduce the different sizes of pipe in series to an equivalent diameter of the given overall length. With this diameter and length, determine Q from the given loss of head. Using this Q, check summation of losses in the line. A pipe diagram is of much assistance in the application of the method.

Example 9.1

A 5 cm pipe takes off abruptly from a large tank, runs 8 m, and then expands abruptly to 10 cm diameter, again runs 45 m, and then discharges directly into the open air with a velocity of 1.5 m/s. Compute the necessary height of water surface above the point of discharge. Take $f_1 = 0.024$ and $f_2 = 0.026$.

Solution We have, $d_1 = 5$ cm $= 0.05$ m, $L_1 = 8$ m, $d_2 = 10$ cm $= 0.10$ m, $L_2 = 45$ m, $V_2 = 1.5$ m/s, $f_1 = 0.024$, $f_2 = 0.026$

Now, height of water surface above the point of discharge = total head loss

$$\therefore H = 0.5\frac{V_1^2}{2g} + f_1\frac{L_1}{d_1}\frac{V_1^2}{2g} + \frac{(V_1 - V_2)^2}{2g} + f_2\frac{L_2}{d_2}\frac{V_2^2}{2g} + \frac{V_2^2}{2g} \qquad \text{(i)}$$

Since the flow is continuous, $Q = A_1V_1 = A_2V_2$

$$\therefore \frac{\pi}{4}d_1^2V_1 = \frac{\pi}{4}d_2^2V_2$$

$$\therefore V_1 = V_2\left(\frac{d_2}{d_1}\right)^2 = 1.5\times\left(\frac{0.10}{0.05}\right)^2 = 6\ m/s$$

$\therefore$ From (i), we have

$$H = 0.5\times\frac{6^2}{2\times 9.81} + 0.024\times\frac{8}{0.05}\times\frac{6^2}{2\times 9.81} + \frac{(6-1.5)^2}{2\times 9.81} + 0.026\times\frac{45}{0.10}\times\frac{1.5^2}{2\times 9.81} + \frac{1.5^2}{2\times 9.81}$$

$$= 0.917 + 7.045 + 1.032 + 1.341 + 0.115 = 10.45\ \text{m}$$

Example 9.2

A pipeline 40 m long is connected to a water tank at one end and discharges freely into the atmosphere at the other end. For the first 25 m of its length from the tank, the pipe is 15 cm in diameter and its diameter suddenly enlarges to 30 cm. The height of water level in the tank is 8 m above the center of the pipe. Considering all losses of head which occur, determine the rate of flow. Assume $f = 0.020$ for both the pipes.

Solution We have, $L = 40$ m, $L_1 = 25$ m, $d_1 = 15$ cm $= 0.15$ m, $L_2 = 40 - 25 = 15$ m, $d_2 = 30$ cm $= 0.30$ m, $H = 8$ m, $f = 0.020$

Since the flow is continuous, $Q = A_1V_1 = A_2V_2$

$$\therefore \frac{\pi}{4}d_1^2V_1 = \frac{\pi}{4}d_2^2V_2$$

$$\therefore V_1 = V_2\left(\frac{d_2}{d_1}\right)^2 = V_2\times\left(\frac{0.30}{0.15}\right)^2 = 4V_2$$

Loss of head at the entrance $= 0.5\frac{V_1^2}{2g} = 0.5\times\frac{(4V_2)^2}{2g} = \frac{8V_2^2}{2g}$

Loss of head due to friction in pipe 1 $= f_1\frac{L_1}{d_1}\frac{V_1^2}{2g} = 0.020\times\frac{25}{0.15}\times\frac{(4V_2)^2}{2g} = 26.67\frac{V_2^2}{2g}$

Loss of head due to sudden enlargement $= \frac{(V_1 - V_2)^2}{2g} = \frac{(4V_2 - V_2)^2}{2g} = \frac{9V_2^2}{2g}$

Loss of head due to friction in pipe 2 $= f_2 \frac{L_2}{d_2}\frac{V_2^2}{2g} = 0.020 \times \frac{15}{0.30} \times \frac{V_2^2}{2g} = \frac{V_2^2}{2g}$

Since the total head available is lost in all the above losses, so

$$8 = \frac{8V_2^2}{2g} + \frac{26.67V_2^2}{2g} + \frac{9V_2^2}{2g} + \frac{V_2^2}{2g} + \frac{V_2^2}{2g} = \frac{45.67V_2^2}{2g}$$

$$\therefore V_2 = \sqrt{\frac{8 \times 2 \times 9.81}{45.67}} = 1.854 \ m/s$$

$$\therefore Q = A_2 V_2 = \frac{\pi}{4} \times 0.30^2 \times 1.854 = 0.131 \ m^3/s$$

9.3 EQUIVALENT PIPES

Two pipe systems are said to be equivalent when the same head loss produces the same discharge in both systems. We have, for the first pipe

$$h_{f1} = f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} = f_1 \frac{L_1}{d_1}\frac{Q^2}{2g(\pi d_1^2/4)^2} = \frac{8Q^2 f_1 L_1}{\pi^2 g d_1^5} \qquad (9.6)$$

and for the second pipe

$$h_{f2} = \frac{8Q^2 f_2 L_2}{\pi^2 g d_2^5} \qquad (9.7)$$

For the two pipes to be equivalent, $h_{f1} = h_{f2}$ for the same discharge Q. So, equating Eqs.(9.6) and (9.7), we have after simplifying

$$\frac{f_1 L_1}{d_1^5} = \frac{f_2 L_2}{d_2^5} \qquad (9.8)$$

$$\therefore L_2 = L_1 \frac{f_1}{f_2}\left(\frac{d_2}{d_1}\right)^5 \qquad (9.9)$$

which determines the length of the second pipe to be equivalent to the first pipe.

Obviously, a compound pipe may be replaced by a pipe of uniform diameter and of the same length as that of a compound pipe, such that the loss of head and discharge are the same in both the cases. The new pipe of uniform diameter is called *equivalent pipe* and its diameter is called *equivalent size of the pipe*.

Let L_1, L_2 and L_3 be the lengths and d_1, d_2 and d_3 be the diameters of the three pipes of a compound pipe. Assume that the three pipes have the same friction factor f as that of the compound pipe. Then, the total head loss in the compound pipe is given by

$$H = f\frac{L_1 V_1^2}{d_1 2g} + f\frac{L_2 V_2^2}{d_2 2g} + f\frac{L_3 V_3^2}{d_3 2g} = \frac{8Q^2 f}{\pi^2 g}\left(\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right) \qquad (9.10)$$

If L is the length and d is the diameter of the equivalent pipe that would give the same discharge for the same head loss due to friction, then total head loss in the equivalent pipe

$$H = \frac{8Q^2 fL}{\pi^2 g d^5} \qquad (9.11)$$

where $L = L_1 + L_2 + L_3$ = length of the equivalent pipe. Equating Eqs.(9.10) and (9.11) and simplifying, we have

$$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} \qquad (9.12)$$

Equation (9.12) gives the uniform diameter of an equivalent pipe which has the same length, same head loss and same discharge as the compound pipe.

It is also sometimes required to determine the length L of a pipe of uniform diameter d which is available for use to replace the compound pipe. To have the same discharge and same loss of head due to friction, the length of such a pipe would be

$$L = d^5\left(\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right) \qquad (9.13)$$

Remember that in this case, $L \neq L_1 + L_2 + L_3$.

Example 9.3

Replace 300 m of 250 mm diameter pipe (f = 0.020) with an equivalent length of 150 mm diameter pipe (f = 0.018).

Solution We have, $L_1 = 300$ m, $d_1 = 250$ mm = 0.25 m, $f_1 = 0.020$, $d_2 = 150$ mm = 0.15 m, $f_2 = 0.018$

From Eq. (9.9), we have

$$L_2 = L_1\frac{f_1}{f_2}\left(\frac{d_2}{d_1}\right)^5 = 300\times\frac{0.020}{0.018}\times\left(\frac{0.15}{0.25}\right)^5 = 25.9\ m$$

Therefore, 25.9 m of 150 mm diameter pipe is equivalent to 300 m of 250 mm diameter pipe.

Example 9.4

A compound pipe line is made up of pipes 45 cm in diameter for 900 m, 37.5 cm in diameter for 450 m and 30 cm in diameter for 300 m. It is required to be replaced by a pipe of uniform diameter. Find the diameter of the new pipe, assuming the length to remain the same.

Solution We have, $d_1 = 45$ cm = 0.45 m, $L_1 = 900$ m, $d_2 = 37.5$ cm = 0.375 m, $L_2 = 450$ m, $d_3 = 30$ cm = 0.30 m, $L_3 = 300$ m and $L = L_1 + L_2 + L_3 = 900 + 450 + 300 = 1650$ m

∴ Using the equation

$$\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$$

we obtain

$$\frac{1650}{d^5} = \frac{900}{0.45^5} + \frac{450}{0.375^5} + \frac{300}{0.30^5} = 232911$$

$$\therefore d = \left(\frac{1650}{232911}\right)^{1/5} = 0.3716\ m = 37.16\ cm$$

9.4 PIPES IN PARALLEL

A combination of two or more pipes connected as in Fig. 9.3, so that the flow is divided among the pipes and then is joined again, is a parallel-pipe system. **In series pipes, the discharge is same in all the pipes and head losses are cumulative. But in parallel pipes, the head losses are the same and discharges are cumulative.**

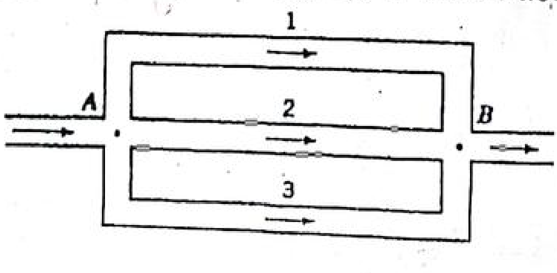

Fig. 9.3 Parallel pipe system

To solve a problem involving parallel-pipe system, the following two conditions must be satisfied:

1. The head loss in each parallel pipe is the same, i.e. $h_{f1} = h_{f2} = h_{f3}$.

2. The discharge in the main line is equal to the sum of the discharges in the parallel lines, i.e. $Q = Q_1 + Q_2 + Q_3$.

A mathematical determination of the division of flow can be made by use of the Darcy-Weisbach formula

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = f\frac{L}{d}\frac{Q^2}{2g(\pi d^2/4)^2} = \frac{8Q^2 fL}{\pi^2 g d^5} \tag{9.14}$$

so that

$$Q = \pi\sqrt{g}\frac{\sqrt{h_f}\, d^{5/2}}{\sqrt{f}\sqrt{L}} \qquad Q = \sqrt{\frac{h_f \pi^2 g d^5}{8 f L}} \tag{9.15}$$

If only two branches 1 and 2 are in parallel, then for a given h_f

$$\frac{Q_1}{Q_2} = \left(\frac{d_1}{d_2}\right)^{5/2} \times \sqrt{\frac{f_2}{f_1}} \times \sqrt{\frac{L_2}{L_1}} \tag{9.16}$$

If the diameters and the lengths of the two branches are known and the friction factors are known or assumed, then Eq. (9.16) can be expressed as

$$Q_1 = FQ_2 \tag{9.17}$$

where F is a known quantity. Moreover,

$$Q = Q_1 + Q_2 \tag{9.18}$$

where Q is the discharge through the approach or the exit pipe. With Q known or assumed, simultaneous solution of Eqs.(9.17) and (9.18) gives Q_1 and Q_2. Then, using these discharges, head losses in pipes 1 and 2 can be computed. These must be equal. If the computations do not show them equal, the discharges should be adjusted by trial until reasonable agreement is obtained.

This method can be extended to any number of branches in parallel. For example, with three branches in parallel as in Fig. 9.3, it is possible to develop the relations

$$Q_2 = F'Q_1 \quad \text{and} \quad Q_3 = F''Q_1 \tag{9.19}$$

which can be combined with the equation

$$Q = Q_1 + Q_2 + Q_3 \tag{9.20}$$

to obtain the flow in each of the three parallel branches.

Two types of problems with parallel-pipe system generally arise:

1. Having given the lengths and the diameters of all the pipes and the allowable head loss, to determine Q.

2. Having given the total discharge Q and the diameters and the lengths of the pipes, to find the distribution of flow and the head loss.

The first type is, in effect, the solution of simple pipe problems for discharge, since the head loss is known. These discharges are added to determine the total discharge,

The second type of problem is more complex, as neither the head loss nor the discharge for any pipe is known. Two methods of solution of this problem are as follows.

Method 1

1. Determine the division of flow in the branches in parallel. To do so, express the discharges Q_2 and Q_3 in terms of Q_1. Then, using Eq. (9.20), determine Q_1. From Q_1, Q_2 and Q_3 can be obtained.

2. Compute the head losses h_{f1}, h_{f2} and h_{f3}. The loss of head in the parallel branches should be equal.

Method 2

1. Assume a discharge Q_1' through pipe 1.

2. Determine h_{f1}' using the assumed discharge.

3. Using h_{f1}', determine Q_2' and Q_3'.

4. The given total discharge Q is split up among the three pipes in the same proportion as Q_1', Q_2' and Q_3', i.e.

$$Q_1 = \frac{Q_1'}{\sum Q'}Q, \quad Q_2 = \frac{Q_2'}{\sum Q'}Q, \quad Q_3 = \frac{Q_3'}{\sum Q'}Q \tag{9.21}$$

where $\sum Q' = Q_1' + Q_2' + Q_3'$.

5. Check the correctness of these discharges by computing h_{f1}, h_{f2} and h_{f3}.

Example 9.5

In Fig. 9.3, L_1 = 900 m, d_1 = 300 mm, f_1 = 0.021, L_2 = 600 m, d_2 = 200 mm, f_2 = 0.018, L_3 = 1200 m, d_3 = 400 mm, f_3 = 0.019 and the head loss between A and B is 6.62 m. Determine the discharge.

Solution From Eq.(9.6), we have for pipe 1

$$h_{f1} = f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} = f_1 \frac{L_1}{d_1}\frac{Q_1^2}{2g(\pi d_1^2/4)^2} = \frac{8Q_1^2 f_1 L_1}{\pi^2 g d_1^5}$$

or

$$Q_1 = \sqrt{\frac{h_{f1} \times \pi^2 \times g \times d_1^5}{8 f_1 L_1}} = \sqrt{\frac{6.62 \times \pi^2 \times 9.81 \times 0.30^5}{8 \times 0.021 \times 900}} = 0.1015\ m^3/s$$

Similarly

$$Q_2 = \sqrt{\frac{h_{f2} \times \pi^2 \times g \times d_2^5}{8 f_2 L_2}} = \sqrt{\frac{6.62 \times \pi^2 \times 9.81 \times 0.20^5}{8 \times 0.018 \times 600}} = 0.0487\ m^3/s$$

$$Q_3 = \sqrt{\frac{h_{f3} \times \pi^2 \times g \times d_3^5}{8 f_3 L_3}} = \sqrt{\frac{6.62 \times \pi^2 \times 9.81 \times 0.40^5}{8 \times 0.019 \times 1200}} = 0.1897\ m^3/s$$

$$\therefore Q = Q_1 + Q_2 + Q_3 = 0.1015 + 0.0487 + 0.1897 = 0.3399\ m^3/s$$

Example 9.6

In Fig. 9.3, L_1 = 900 m, d_1 = 300 mm, f_1 = 0.021, L_2 = 600 m, d_2 = 200 mm, f_2 = 0.018, L_3 = 1200 m, d_3 = 400 mm and f_3 = 0.019. For a discharge of 0.34 m³/s, determine the flow through each pipe and the loss of head between A and B.

Solution by Method 1 We have from Eq.(9.16)

$$\frac{Q_2}{Q_1} = \left(\frac{d_2}{d_1}\right)^{5/2} \times \sqrt{\frac{f_1}{f_2}} \times \sqrt{\frac{L_1}{L_2}} = \left(\frac{0.20}{0.30}\right)^{5/2} \times \sqrt{\frac{0.021}{0.018}} \times \sqrt{\frac{900}{600}} = 0.480$$

$\therefore Q_2 = 0.480Q_1$ (i)

Similarly

$$\frac{Q_3}{Q_1} = \left(\frac{d_3}{d_1}\right)^{5/2} \times \sqrt{\frac{f_1}{f_3}} \times \sqrt{\frac{L_1}{L_3}} = \left(\frac{0.40}{0.30}\right)^{5/2} \times \sqrt{\frac{0.021}{0.019}} \times \sqrt{\frac{900}{1200}} = 1.869$$

$\therefore Q_3 = 1.869Q_1$ (ii)

Since $Q = Q_1 + Q_2 + Q_3$, we have

$0.34 = Q_1 + 0.480Q_1 + 1.869Q_1 = 3.349Q_1$

$$\therefore Q_1 = \frac{0.34}{3.349} = 0.1015\ m^3/s$$

Then

$$Q_2 = 0.480 \times 0.1015 = 0.0487\ m^3/s$$

and

$$Q_3 = 1.869 \times 0.1015 = 0.1897\ m^3/s$$

Now, check the values of h_{f1}, h_{f2} and h_{f3}. Using Eq.(9.6), we have

$$h_{f1} = \frac{8Q_1^2 f_1 L_1}{\pi^2 g d_1^5} = \frac{8 \times 0.1015^2 \times 0.021 \times 900}{\pi^2 \times 9.81 \times 0.30^5} = 6.621\ m$$

Similarly

$$h_{f2} = \frac{8Q_2^2 f_2 L_2}{\pi^2 g d_2^5} = \frac{8 \times 0.0487^2 \times 0.018 \times 600}{\pi^2 \times 9.81 \times 0.20^5} = 6.614\ m$$

$$h_{f3} = \frac{8Q_3^2 f_3 L_3}{\pi^2 g d_3^5} = \frac{8 \times 0.1897^2 \times 0.019 \times 1200}{\pi^2 \times 9.81 \times 0.40^5} = 6.620\ m$$

It is seen that the loss of head in the three branches is almost the same.

Solution by Method 2 Assume $Q_1' = 0.085$ m³/s. Then, using Eq.(9.6)

$$h_{f1}' = \frac{8Q_1'^2 f_1 L_1}{\pi^2 g d_1^5} = \frac{8 \times 0.085^2 \times 0.021 \times 900}{\pi^2 \times 9.81 \times 0.30^5} = 4.643\ m$$

$\therefore$ For pipe 2, we get

$$Q_2' = \sqrt{\frac{h_{f1}' \times \pi^2 \times g \times d_2^5}{8 f_2 L_2}} = \sqrt{\frac{4.643 \times \pi^2 \times 9.81 \times 0.20^5}{8 \times 0.018 \times 600}} = 0.0408\ m^3/s$$

and for pipe 3, we obtain

$$Q_3' = \sqrt{\frac{h_{f1} \times \pi^2 \times g \times d_3^5}{8 f_3 L_3}} = \sqrt{\frac{4.643 \times \pi^2 \times 9.81 \times 0.40^5}{8 \times 0.019 \times 1200}} = 0.1589\ m^3/s$$

The total discharge for the assumed condition is

$Q_1' + Q_2' + Q_3' = 0.085 + 0.0408 + 0.1589 = 0.2847\ m^3/s$

But the total discharge is 0.34 m³/s. So correcting the discharges in proportion, we obtain

$$Q_1 = \frac{0.085}{0.2847} \times 0.34 = 0.1015\ m^3/s$$

$$Q_2 = \frac{0.0408}{0.2847} \times 0.34 = 0.0487\ m^3/s$$

$$Q_3 = \frac{0.1589}{0.2847} \times 0.34 = 0.1898\ m^3/s$$

Now, check the values of h_{f1}, h_{f2} and h_{f3} using Eq.(9.6).

$$h_{f1} = \frac{8Q_1^2 f_1 L_1}{\pi^2 g d_1^5} = \frac{8 \times 0.1015^2 \times 0.021 \times 900}{\pi^2 \times 9.81 \times 0.30^5} = 6.621\ m$$

$$h_{f2} = \frac{8Q_2^2 f_2 L_2}{\pi^2 g d_2^5} = \frac{8 \times 0.0487^2 \times 0.018 \times 600}{\pi^2 \times 9.81 \times 0.20^5} = 6.614\ m$$

$$h_{f3} = \frac{8Q_3^2 f_3 L_3}{\pi^2 g d_3^5} = \frac{8 \times 0.1898^2 \times 0.019 \times 1200}{\pi^2 \times 9.81 \times 0.40^5} = 6.627\ m$$

It is seen that the loss of head in the three branches is almost the same.

Note: Method 1 seems to be better than method 2.

9.5 PIPE NETWORKS

City water supply distribution systems are constructed in the form of many loops, branches and junctions, more or less complicated in arrangement. Such a system is called a network. The methods we have so far considered are not suitable for solving complicated pipe network problems. An important advance in the solution of pipe network problems was made by Hardy Cross, who developed a method of successive approximations by which the distribution of flow in a pipe network can be determined.

Consider an elementary loop A in a general network of pipes, as illustrated in Fig. 9.4, The arrowheads indicate direction of flow. The following two conditions must be satisfied for the solution of pipe network problems.

1. At any junction the total inflow must be equal to the total outflow.

2. The loss of head due to flow in a clockwise direction around a loop must be equal to the loss of head due to flow in a counterclockwise direction. Thus in Fig. 9.4, the loss of head in pipes ab and bc must be equal to the loss of head in pipes ad, de and ec.

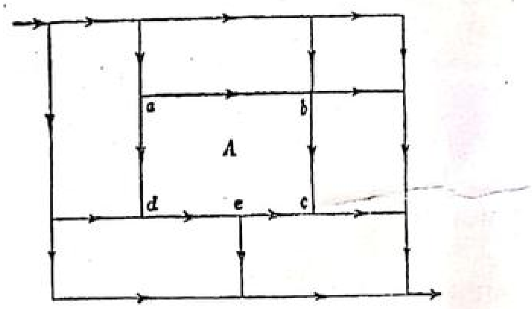

Fig. 9.4 Pipe network

A flow may be assumed in each pipe of the loop which will meet the first condition. Such assumed flow would meet condition 2 only by good luck. The computed loss of head in the clockwise flow will not ordinarily be equal to the loss of head in counterclockwise flow. Hardy Cross developed the following method of computing a correction to the assumed flow that would tend to equalize the loss of head in the two directions.

For a given size, length and roughness of pipe, the loss of head varies as some power of the discharge, or

$$h_f = KQ^n \tag{9.22}$$

The values of K and n depend on the formula used. For the Darcy-Weisbach formula, it can be shown that

$$K = \frac{8fL}{\pi^2 g d^5} \tag{9.23}$$

and

$$n = 2 \tag{9.24}$$

Let the symbols $\sum_c$ and $\sum_{cc}$ denote the summation of quantities in the clockwise and anticlockwise directions, respectively. In any elementary loop A, the loss of head in clockwise flow is the sum of the losses in all pipes in which flow is clockwise around the loop, and can be expressed as

$$\textstyle\sum_c h_f = \sum_c K Q_c^n \tag{9.25}$$

Likewise, the loss of head in counterclockwise flow can be expressed as

$$\sum_{cc} h_f = \sum_{cc} KQ_{cc}^{\,n} \qquad (9.26)$$

As pointed out above, the first assumed direction of flow will ordinarily not result in equality of $\sum_c h_f$ and $\sum_{cc} h_f$. Assuming $\sum_c h_f$ to be larger, the positive quantity given by the expression

$$\sum_c KQ_c^{\,n} - \sum_{cc} KQ_{cc}^{\,n}$$

represents the "error" of the loss of head. It is desired to determine the amount of the flow correction ΔQ which, when subtracted from Q_c and added to Q_{cc}, will equalize the head losses in the two directions, and satisfy the condition

$$\sum_c K(Q_c - \Delta Q)^n = \sum_{cc} K(Q_{cc} + \Delta Q)^n \qquad (9.27)$$

Expanding the quantities in parenthesis by the binomial theorem and retaining only the first two terms

$$\sum_c K(Q_c^{\,n} - nQ_c^{\,n-1}\Delta Q) = \sum_{cc} K(Q_{cc}^{\,n} + nQ_{cc}^{\,n-1}\Delta Q) \qquad (9.28)$$

Solving for ΔQ, we obtain

$$\Delta Q = \frac{\sum_c KQ_c^n - \sum_{cc} KQ_{cc}^n}{n\left(\sum_c KQ_c^{n-1} + \sum_{cc} KQ_{cc}^{n-1}\right)} \qquad (9.29)$$

From Eq.(9.22), dividing by Q

$$KQ^{n-1} = \frac{h_f}{Q} \qquad (9.30)$$

Using Eqs.(9.25), (9.26) and (9.30) into Eq. (9.29), we get

$$\Delta Q = \frac{\sum_c h_f - \sum_{cc} h_f}{n\left(\sum_c \frac{h_f}{Q} + \sum_{cc} \frac{h_f}{Q}\right)} \qquad (9.31)$$

Example 9.7

Determine the distribution of flow in the pipe network shown in Fig. 9.5(a). The flows are in liters/sec. Use the Hardy Cross method.

Solution

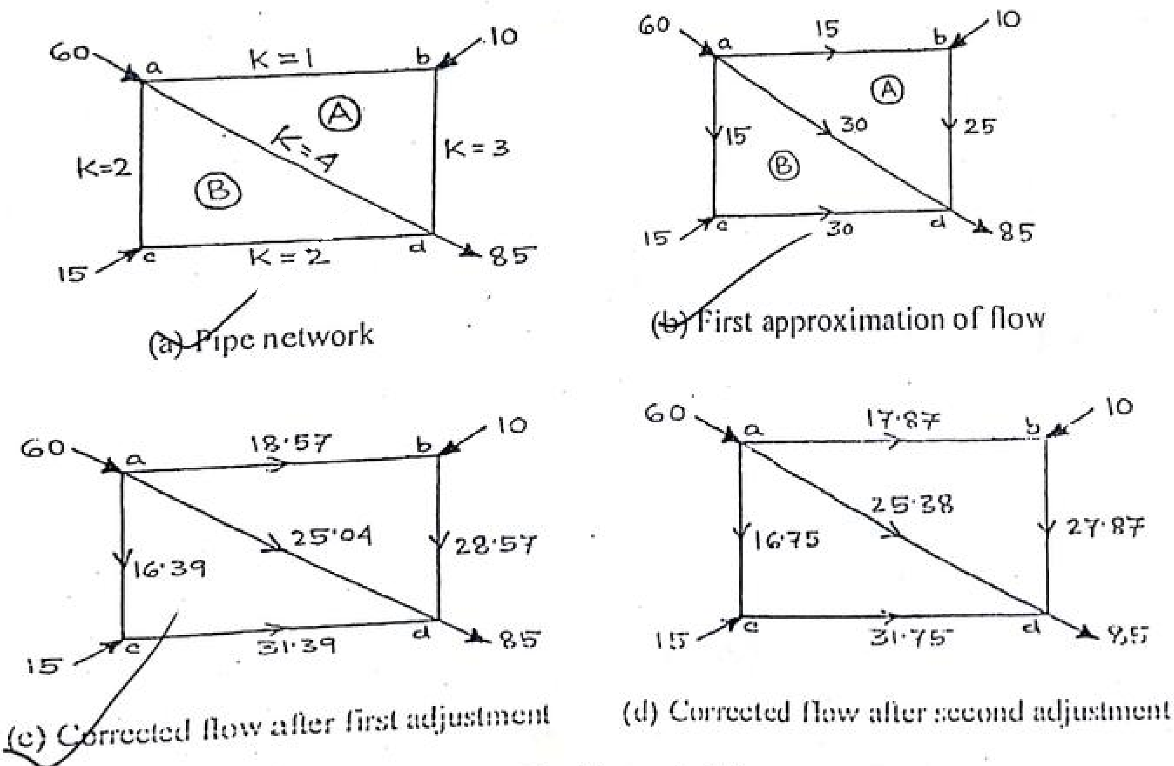

Fig. 9.5 for Example 9.7

An assumption of the flows in all pipes is made as shown in Fig. 9.5(b) such that the total inflow equals the total outflow at each junction. With n = 2 and the given K-values and using Eq.(9.22), the loss of head in the pipes in a clockwise direction and in a counterclockwise direction is computed for the loops A and B and the sum of the losses in each direction is determined. The quantity h_f/Q is also computed for all pipes in each loop and summed up. Equation (9.31) is then used to determine ΔQ for each loop. The computations are shown in a table in which the clockwise losses are shown first for each loop.

The corrections are then applied to the flows of Fig. 9.5(b). If the clockwise losses exceed the counterclockwise losses, the algebraic sum of their difference is positive and the clockwise flow must be reduced by an amount ΔQ and the counterclockwise flow increased by the same amount. Pipe ad is common to two loops and therefore requires a double correction. The corrected flow after the first adjustment is shown in Fig. 9.5(c). A second computation using the corrected flows is then made and the corrected flow after the second adjustment is shown in Fig. 9.5(d). The process is repeated till the corrections become negligible in amount.

First adjustment

Loop A

Pipe	K	Assumed Q	$h_f = KQ^2$	h_f/Q
ab	1	15	225	15
bd	3	25	1875	75
			2100	
ad	4	30	3600	120
			3600	210

$$\Delta Q = \frac{2100 - 3600}{2 \times 210} = -3.57$$

Loop B

Pipe	K	Assumed Q	$h_f = KQ^2$	h_f/Q
ad	4	26.43	2794.18	105.72
			2794.18	
ac	2	15	450	30
cd	2	30	1800	60
			2250	195.72

$$\Delta Q = \frac{2794.18 - 2250}{2 \times 195.72} = +1.39$$

Second adjustment

Loop A

Pipe	K	Assumed Q	$h_f = KQ^2$	h_f/Q
ab	1	18.57	344.84	18.57
bd	3	28.57	2448.73	85.71
			2793.57	
ad	4	25.04	2508.01	100.16
			2508.01	204.44

$$\Delta Q = \frac{2793.57 - 2508.01}{2 \times 204.44} = +0.70$$

Loop B

Pipe	K	Assumed Q	$h_f = KQ^2$	h_f/Q
ad	4	25.74	2650.19	102.96
			2650.19	
ac	2	16.39	537.26	32.78
cd	2	31.39	1970.66	62.78
			2507.92	198.52

$$\Delta Q = \frac{2650.19 - 2507.92}{2 \times 198.52} = +0.36$$

9.6 BRANCHING PIPES CONNECTING RESERVOIRS AT DIFFERENT ELEVATIONS: THE THREE-RESERVOIR PROBLEM

In Fig. 9.6, A, B and C are three reservoirs connected by pipes 1, 2 and 3. A condition of steady flow with constant reservoir levels is assumed. Let L_1, d_1, Q_1 and V_1 represent respectively the length, diameter, discharge and mean velocity for pipe 1, and the same symbols with subscripts 2 and 3 represent the corresponding terms for pipes 2 and 3. Assume that reservoir A supplies reservoirs B and C so that $Q_1 = Q_2 + Q_3$. Many problems with branching pipes connecting reservoirs arise. Methods of solving three of these problems are outlined.

1. Having given the lengths and diameters of all pipes and elevations of the three reservoirs, to determine Q_1, Q_2 and Q_3.

This problem can be solved analytically. Using the Darcy-Weisbach formula, from Fig 9.6

$$H_B = h_{f1} + h_{f2} = f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} + f_2 \frac{L_2}{d_2}\frac{V_2^2}{2g} \tag{9.32}$$

and

$$H_C = h_{f1} + h_{f3} = f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} + f_3 \frac{L_3}{d_3}\frac{V_3^2}{2g} \tag{9.33}$$

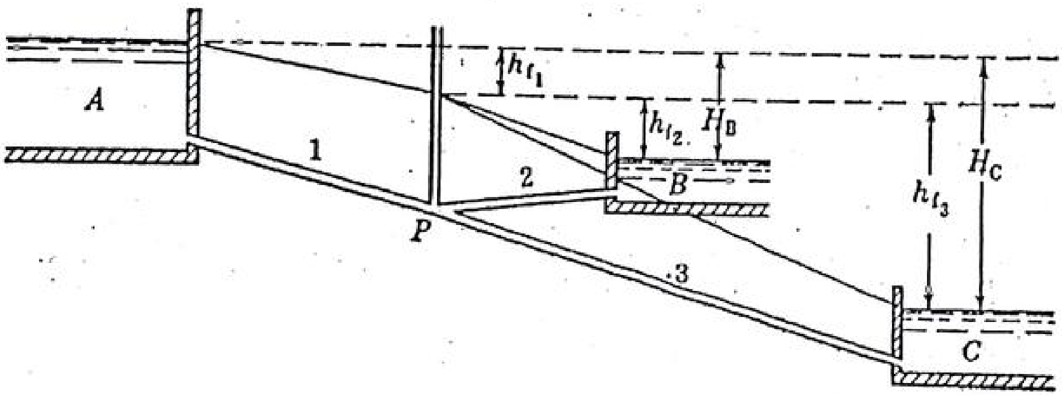

Fig. 9.6 Branching pipes connecting three reservoirs

Also, since $Q_1 = Q_2 + Q_3$, we have

$$d_1^2 V_1 = d_2^2 V_2 + d_3^2 V_3 \tag{9.34}$$

With H_B, H_C, the lengths and diameters of all pipes known, the above three equations can be solved simultaneously for V_1, V_2 and V_3.

2. Having given the lengths and diameters of all pipes, Q_1 and the elevations of water surfaces in reservoirs A and B, to determine the elevation of water surface in reservoir C.

Using Q_1, determine h_{f1}, the head loss in pipe 1. Then, $h_{f2} = H_B - h_{f1}$, the head loss in pipe 2, using which Q_2 can be determined. Then $Q_3 = Q_1 - Q_2$, With Q_3, the head loss in pipe 3 can be computed and the elevation of water surface in reservoir C can be obtained.

3. Having given the lengths of all pipes, the elevations of water surfaces in all reservoirs, Q_1 and the diameters of two pipes d_1 and d_2, to determine d_3.

Determine h_{f1}, Q_2 and Q_3 as for case 2. Then with Q_3 and h_{f3} = H_C - h_{f2} known, compute d_3.

Example 9.8

A 60 cm pipe is supplied with water from a reservoir A (Fig. 9.6), and at a point P it is divided into two branches of 45 cm and 30 cm diameter, which discharge into reservoirs B and C, respectively. Length of 60 cm pipe is 600 m, length of 45 cm pipe is 900 m and length of 30 cm pipe is 450 m. The surface levels in A, B and C and the level at P are 30, 21, 15 and 24 m above datum, respectively. Find the velocity of flow and discharge in each pipe. Take f = 0.028. Neglect losses other than due to friction.

Solution We have, d_1 = 60 cm = 0.60 m, d_2 = 45 cm = 0.45 m, d_3 = 30 cm = 0.30 m, L_1 = 600 m, L_2 = 900 m, L_3 = 450 m, z_A = 30 m, z_B = 21 m, z_C = 15 m and z_P = 24 m and f = 0.028. Referring to Fig. 9.6

$$H_B = z_A - z_B = h_{f1} + h_{f2} = f\frac{L_1}{d_1}\frac{V_1^2}{2g} + f\frac{L_2}{d_2}\frac{V_2^2}{2g}$$

or, $$30-21 = 0.028\times\frac{600}{0.60}\times\frac{V_1^2}{2\times 9.81} + 0.028\times\frac{900}{0.45}\times\frac{V_2^2}{2\times 9.81}$$

or, $$9 = 28\frac{V_1^2}{2g} + 56\frac{V_2^2}{2g}$$

$$\therefore V_2 = \sqrt{3.153 - 0.5V_1^2} \qquad \text{(i)}$$

Again, $$H_C = z_A - z_C = h_{f1} + h_{f3} = f\frac{L_1}{d_1}\frac{V_1^2}{2g} + f\frac{L_3}{d_3}\frac{V_3^2}{2g}$$

or, $$30-15 = 0.028\times\frac{600}{0.60}\times\frac{V_1^2}{2\times 9.81} + 0.028\times\frac{450}{0.30}\times\frac{V_3^2}{2\times 9.81}$$

or, $$15 = 28\frac{V_1^2}{2g} + 42\frac{V_3^2}{2g}$$

$$\therefore V_3 = \sqrt{7.007 - 0.667V_1^2} \qquad \text{(ii)}$$

Since $Q_1 = Q_2 + Q_3$ or $d_1^2V_1 = d_2^2V_2 + d_3^2V_3$, we have

$$0.60^2 \times V_1 = 0.45^2 \times V_2 + 0.30^2 \times V_3$$

or

$$0.36V_1 = 0.2025\sqrt{3.153 - 0.5V_1^2} + 0.09\sqrt{7.007 - 0.667V_1^2}$$

By trial, we obtain

$$V_1 = 1.42 \text{ m/s}$$

Then

$$V_2 = \sqrt{3.153 - 0.5\times 1.42^2} = 1.46\ m/s$$

and

$$V_3 = \sqrt{7.007 - 0.667\times 1.42^2} = 2.38\ m/s$$

Therefore, we obtain

$$Q_1 = \frac{\pi}{4}\times 0.60^2 \times 1.42 = 0.40\ m^3/s$$

$$Q_2 = \frac{\pi}{4}\times 0.45^2 \times 1.46 = 0.23\ m^3/s$$

$$Q_3 = \frac{\pi}{4}\times 0.30^2 \times 2.38 = 0.17\ m^3/s$$

Check: $Q_2 + Q_3 = 0.23 + 0.17 = 0.40\ m^3/s = Q_1$

PROBLEMS AND EXERCISES

9.1 State when two pipe systems are said to be equivalent.

9.2 What do you mean by equivalent pipe?

9.3 State the conditions of discharges and head losses when the pipes are (i) connected in series, and (ii) connected in parallel.

9.4 What is a pipe network? State the two conditions which are to be satisfied for solving a pipe network problem.

9.5 Describe the Hardy Cross method to determine the distribution flow in a pipe network.

9.6 Suppose you want to increase the discharge through a long pipe conveying water from one reservoir to another reservoir under the same head of water. How can you do it?

9.7 Water is discharged from a tank to another tank having 30 m difference of water levels through a pipe 1200 m long. The diameter of the first 600 m length of the pipe is 400 mm, for the next 400 m length of the pipe is 300 mm and for the remaining length of the pipe is 250 mm. Find the discharge through the pipe system, taking into consideration the frictional losses only. Assume f = 0.020.

9.8 A 50 mm pipe takes off abruptly from a large tank, runs 20 m and then expands abruptly to 100 mm diameter, again runs for 30 m and then discharges directly into the open air. The discharge through the pipe system is 10 liters/sec. Compute the necessary height of water surface above the point of discharge. Take $f_1 = 0.020$ and $f_2 = 0.024$. Include the minor losses.

9.9 (a) Replace 500 m of 200 mm diameter pipe (f = 0.016) with an equivalent length of 100 mm diameter pipe (f = 0.020).

(b) A compound pipe consists of three pipes connected in series. The first pipe is 900 m long and 45 cm in diameter. The second pipe is 450 m long and 40 cm in diameter. The third pipe is 300 m long and 30 cm in diameter. Find the equivalent length of a pipe which has a uniform diameter of 50 cm to replace the compound pipe.

(c) If the compound pipe of Problem 9.9(b) is to be replaced by a pipe of uniform diameter but of the same length, find the diameter of the equivalent pipe.

9.10 Two pipes are connected in parallel between two reservoirs as shown in Fig. 9.7 with length L_1 = 2400 m, d_1 = 1.2 m, f_1 = 0.026, L_2 = 2400 m, d_2 = 1 m and f_2 = 0.019. Find the total discharge, if the difference in water surface elevations in the two reservoirs is 3.5 m.

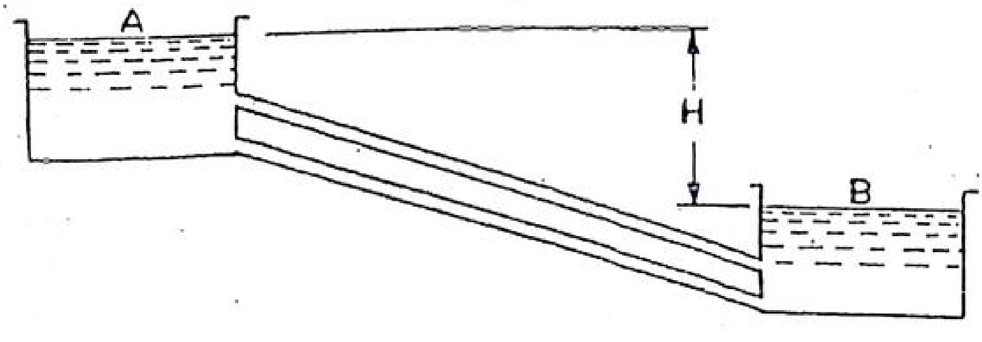

Fig. 9.7 (Problem 9.10)

9.11 If the head loss from A to B in Fig. 9.8 is 4 m, determine the total flow. Take f = 0.020 for both the pipes.

9.12 The discharge of the pipe system shown in Fig. 9.9 is 0.50 m^3/s. Determine the head loss from A to D. Take f = 0.020 for all the pipes.

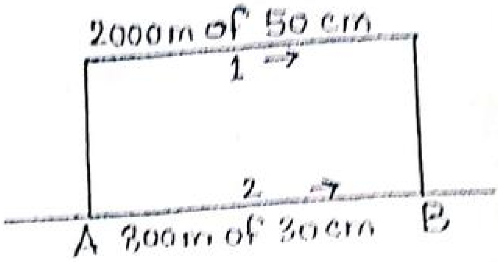

Fig. 9.8 (Problem 9.11)

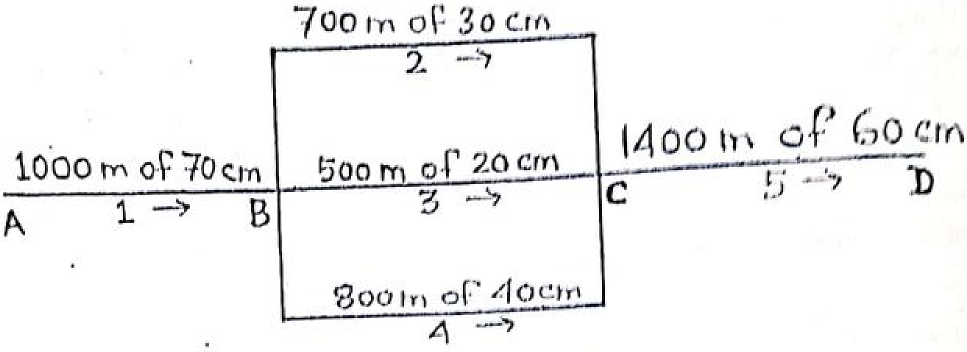

Fig. 9.9 (Problem 9.12)

9.13(a) Two reservoirs having a difference of water levels of 15 m as shown in Fig. 9.10 are connected by a pipe 200 mm in diameter and 3000 m long. Calculate the discharge through the pipe in liters per minute. Take f = 0.024 and neglect all losses other than that due to friction.

(b) If a parallel line 300 mm in diameter and 1200 m long is connected to the last 1200 m of the pipe line to increase the discharge to the lower reservoir, calculate the increase in discharge in liters per minute due to the addition of the parallel line.

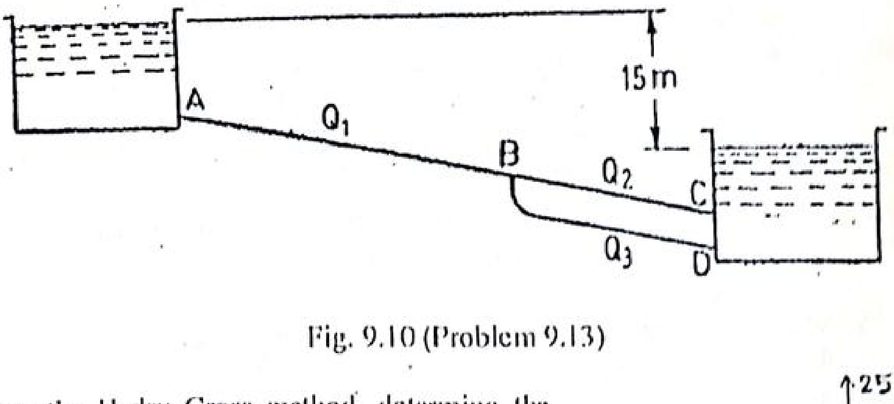

Fig. 9.10 (Problem 9.13)

9.14 Using the Hadry Cross method, determine the flows in the network shown in Fig. 9.11. The flows are in liters/sec.

9.15 Three reservoirs A, B and C are connected by a pipe system as shown in Fig. 9.6. Find the discharge in each pipe if f = 0.027. Pipe AP is 800 m long and 600 mm in diameter, pipe PB is1000 m long and 400 mm in diameter and pipe PC is 500 m long and 300 mm in diameter. The surface levels in A, B and C and the level at P are 32, 22, 18 and 25 m above datum, respectively.

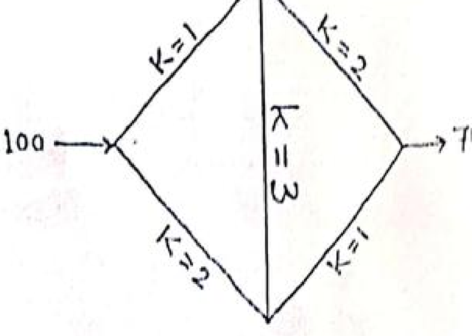

Fig. 9.11 (Problem 9.14)

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Chapter 10

FLUID MEASUREMENT

10.1 INTRODUCTION

Fluid measurements include the determination of pressure, velocity, discharge, shock waves, density gradients, turbulence and viscosity. There are many ways these measurements may be taken, e.g. direct, indirect, gravimetric, volumetric, electronic, electromagnetic and optical. Direct measurements for discharge consist in the determination of the volume or weight of fluid that passes a section in a given time interval. Indirect methods of discharge measurement require the determination of head, difference in pressure, or velocity at several points in a cross-section and, with these, computing the discharge. The most precise methods are the gravimetric or volumetric determinations, in which the weight or volume is measured by weight scales or by a calibrated tank for a time interval that is measured by a stop watch.

10.2 PITOT TUBE

A bent L-shaped glass tube with both ends open, similar to Fig. 10.1, is called a Pitot tube, after the French scientist Henry Pitot (1695--1771). It is used for measuring the velocity of flow at a point in a pipe or stream. When the tube is first placed in a moving stream with one end facing the current, the liquid enters the tube until the liquid surface rises a distance equal to h due to pressure exerted by the flowing liquid. Let point 1 be in the undisturbed stream far enough upstream so that the velocity here is not affected by the presence of the tube. Let point 2 be on the axis of the tube and at the same elevation as point 1. As the liquid particle moves from 1 to 2, its velocity is gradually reduced from v at 1 to practically zero at 2.

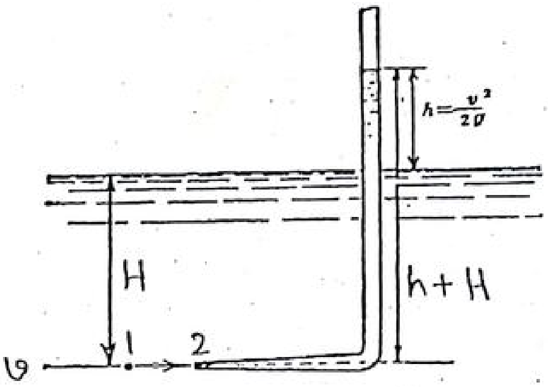


Fig. 10.1 Pitot tube

Applying the energy equation between points 1 and 2, the elevation heads being equal, and neglecting losses

$$H+\frac{v^2}{2g}=(H+h)+0$$

or

$$v=\sqrt{2gh} \qquad (10.1)$$

Hence, the velocity head $v^2/2g$ at 1 is transformed into pressure head at 2, and because of the increased pressure inside the tube, a column of head $v^2/2g$ is maintained inside the tube above the free liquid surface outside. By measuring the rise of liquid h in the tube, we can find out the velocity of flow at point 1.

The velocity obtained by Eq.(10.1) is often multiplied by a coefficient ϕ, known as the *Pitot tube coefficient*, such that

$$v=\phi\sqrt{2gh} \qquad (10.2)$$

This coefficient takes into account the error due to turbulence and energy losses. The numerical value of this coefficient varies from 0.95 to 1.0.

The Pitot tube method is one of the most accurate methods of finding the velocity of flow in a pipe or in an open channel.

The Pitot tube can be used to measure the velocity of water in an open channel as well as in a closed pipe. For an open channel, a simple Pitot tube, explained above, will serve the purpose. However, for a closed pipe in which the water is flowing under pressure, it is necessary to measure the static pressure p also. Then the velocity head will be equal to the total Pitot tube reading minus the static pressure head. The static pressure head is measured by inserting another L-shaped tube with its end pointing towards the flow downstream (Fig. 10.2a). The water will be drawn in this tube due to static pressure. If now the tubes are connected by an inverted U-tube manometer, the difference of water height h will give the velocity head. Such an arrangement is known as *Pitot meter* or *Pitot static tube*. The static pressure can also be measured by inserting the other end of the inverted U-tube to the pipe, as shown in Fig. 10.2(b).

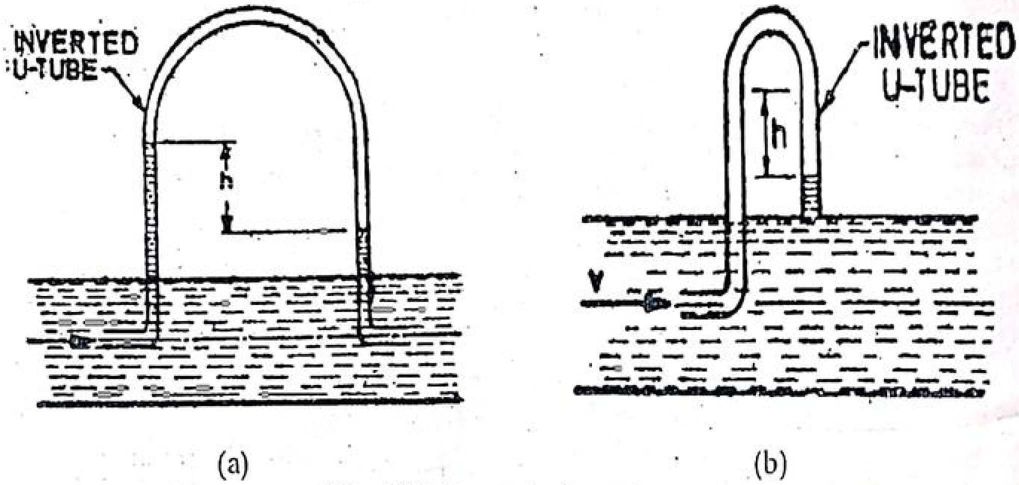


Fig. 10.2 Pitot tube in a pipe

Example 10.1

A Pitot tube is placed in water as shown in Fig. 10.1 and the liquid rises in the tube to a height of 10 cm above the water surface outside the tube. Calculate the velocity of stream upstream of the tube.

Solution Here, h = 10 cm = 0.10 m

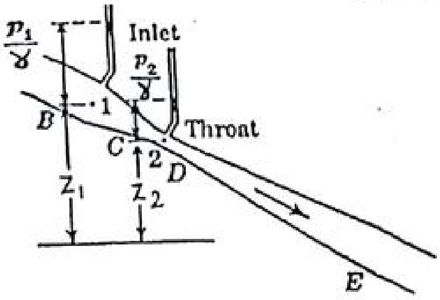
The velocity of the stream upstream of the tube is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.10} = 1.40\ m/s$$

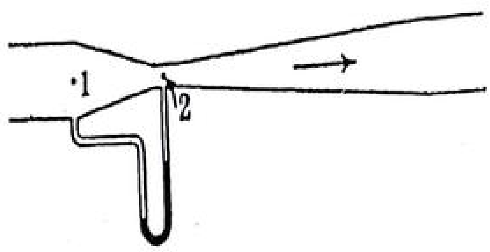
10.3 VENTURIMETER

An example of the practical use of the Bernoulli equation is provided by the Venturimeter. This instrument, which is widely used for measuring discharge through a pipe, especially large discharges, was invented by an American engineer, Clemens Herschel, in 1887 and was named by him in honor of the original discoverer of the principle involved, G. B. Venturi, an Italian philosopher.

A Venturimeter (Fig. 10.3) in its simple form consists of (i) a short converging cone BC connected to the approach pipe at the inlet end B, (ii) a cylindrical throat CD, and (iii) a gradually diverging cone DE connected to the pipe at the outlet end E. The Venturimeter may be horizontal, vertical or inclined, but it is generally kept horizontal. The Venturimeter must run full and it should be preceded by a straight length of pipe of not less than 5 to 10 diameters of pipe to reduce most of the turbulence in order to get accurate results. In order to avoid the tendency of fluid to separate at the throat due to increase in velocity and decrease in pressure, the ratio of the throat diameter to pipe diameter should be ¼ to ¾, the most suitable value being ⅓ to ½. The length of the divergent cone is made equal to 3 to 4 times the length of the convergent cone to reduce the frictional loss of head caused by turbulence as the velocity is reduced. When the frictional losses are of least consideration, the Venturimeter can be made with convergent and divergence angles up to 30° and 14°, respectively. The size of a Venturimeter is expressed in terms of inlet and throat diameters, e.g. a 30 cm × 10 cm Venturimeter fits a 30 cm diameter pipe and has a throat of 10 cm diameter.



(a) Inclined Venturimeter



(b) Horizontal Venturimeter with differential manometer

Fig. 10.3 Venturimeter

Let V_1, p_1 and z_1 represent the mean velocity, pressure and elevation, respectively, at point 1 in the inlet. Let V_2, p_2 and z_2 represent the corresponding quantities at point 2 in the throat. Writing energy equation between points 1 and 2, neglecting friction and assuming uniform velocity in each cross-section, we get

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (10.3)$$

or

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \left(z_1 + \frac{p_1}{\gamma}\right) - \left(z_2 + \frac{p_2}{\gamma}\right) \qquad (10.4)$$

This equation shows that the increase in kinetic energy is equal to the decrease in potential energy, a statement which has been called the *Venturi principle*. The decrease in potential energy is the difference in levels of liquid in piezometer tubes connected to the inlet and throat. It is commonly measured by means of a U-tube or differential manometer connecting inlet and throat (Fig. 10.3b). Two piezometer tubes inserted at the inlet and throat may also serve the purpose (Fig. 10.3a).

Now, applying the equation of continuity between sections 1 and 2

$$Q = A_1V_1 = A_2V_2 \qquad (10.5)$$

Combining Eqs.(10.4) and (10.5), the velocity at either section 1 or 2 can be obtained. With the area known, the theoretical discharge Q_t can be computed. For example, if we compute V_2, then from Eq.(10.5)

$$\frac{V_2^2}{2g} = \frac{V_1^2}{2g}\left(\frac{A_1}{A_2}\right)^2 \qquad (10.6)$$

and combining Eqs.(10.4) and (10.6), we obtain

$$V_2 = \frac{A_1}{\sqrt{A_1^2 - A_2^2}}\sqrt{2gh} \qquad (10.7)$$

so that

$$Q_t = A_2V_2 = \frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2gh} \qquad (10.8)$$

where

$$h = \left(z_1 + \frac{p_1}{\gamma}\right) - \left(z_2 + \frac{p_2}{\gamma}\right) \qquad (10.9)$$

When the Venturimeter is horizontal, $z_1 = z_2$, and

$$h = \frac{p_1}{\gamma} - \frac{p_2}{\gamma} \qquad (10.10)$$

Here, h is known as the Venturi head and represents the difference in pressure heads.

The actual discharge Q of a Venturimeter will be less than the theoretical discharge Q_t given by Eq.(10.8) owing to the non-uniform distribution of velocity over the pipe section as well as friction and other losses. The theoretical discharge is therefore multiplied by a correction factor C_d, less than unity, to get the actual discharge Q, or

$$Q_a = C_d \times Q_t \qquad (10.11)$$

The correction factor C_d is known as the *meter coefficient* or the *discharge coefficient*. It is best determined by measuring the actual flow Q through the Venturimeter by volume or weight, computing the theoretical flow Q_t from the manometer reading and the meter dimensions and computing the ratio of Q to Q_t. The value of C_d is affected by the ratio of A_1 to A_2, velocity, kinematic viscosity of fluid flowing and the roughness of the inner surface of the Venturimeter. However, the coefficient of a standard Venturimeter has a fairly constant value between 0.96 and 0.99.

In Eq.(10.8), the factor $\frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g}$ is constant for a particular Venturimeter and is generally known as *Venturimeter constant*.

Venturimeters are usually installed in an approximately horizontal position. However, for a given discharge the difference between the elevations of the liquids in the two piezometers (Fig. 10.3a) or the differential manometer reading (Fig. 10.3b) will be the same regardless of weather the meter is horizontal or inclined. Since it is assumed that the discharge remains the same, the increase in kinetic energy and likewise the decrease in potential energy must also remain unchanged regardless of the position of the meter.

Example 10.2

A horizontal Ventuimeter having a throat 10 cm in diameter is installed in a 30 cm pipe and is used for measuring the flow of oil of specific gravity 0.9. The oil-mercury differential manometer shows a gauge difference of 20 cm. Calculate the actual discharge in liters per sec if the meter coefficient is 0.98.

Solution We have, $d_1 = 30$ cm $= 0.30$ m, $d_2 = 10$ cm $= 0.10$ m, Specific gravity of oil = 0.9,

Pressure difference = 20 cm, $C_d = 0.98$

Area at inlet, $A_1 = \frac{\pi}{4} \times d_1^2 = \frac{\pi}{4} \times 0.30^2 = 0.071\ m^2$

Area at throat, $A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times 0.10^2 = 0.0079\ m^2$

Difference in pressure head, h = 20 cm of mercury

$$= \frac{20}{100}\left(\frac{13.6}{0.9} - 1\right) = 2.822\ m \text{ of oil}$$

$$\therefore Q_t = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2gh} = \frac{0.071 \times 0.0079}{\sqrt{0.071^2 - 0.0079^2}} \times \sqrt{2 \times 9.81 \times 2.822}$$

$= 0.0588\ m^3/s = 58.80$ liters/sec

$$\therefore Q = C_d \times Q_t = 0.98 \times 58.80 = 57.63 \text{ liters/sec}$$

10.4 ORIFICE

Description

An orifice is an opening in the wall of a vessel through which a liquid flows. The usual purpose of an orifice is the measurement of flow. The orifices are classified according to their size, shape, shape of the upstream end and discharge condition as follows.

a) Size : Small and large orifices
b) Shape : Circular, rectangular, square and triangular orifices
c) Shape of the upstream end: Sharp-edged and rounded or bell-mouthed orifices
d) Discharge condition : Freely discharging and drowned or submerged orifices

Theoretical Velocity

Consider a vessel fitted with a vertical rounded orifice discharging water into the atmosphere under a head of H as shown in Fig. 10.4. The upper surface of the liquid in the vessel is also exposed to the atmosphere.

The liquid particles, in order to flow out through the orifice, move towards the orifice from all directions. Because of the inertia of these particles, they cannot make abrupt changes in their directions to reach the orifice and they therefore follow curvilinear paths that cause the jet to contract for a short distance after leaving the orifice. The section C-C where contraction of the jet is maximum is known as the *vena contracta*. For a sharp-edged orifice, the vena contracta is found to be at a distance of about one-half diameter downstream from the plane of the orifice. At the vena contracta, the streamlines are parallel to one another, the pressure is atmospheric and the velocity of liquid is maximum. Obviously, there is no contraction of the jet beyond the vena contracta,

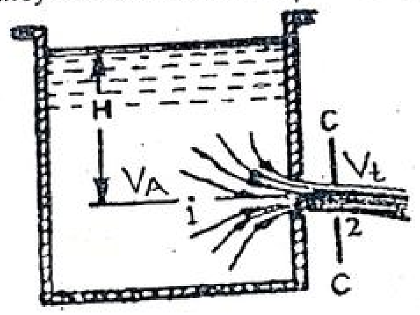


Fig. 10.4 Discharge from orifice

Since liquid particles at different elevations discharge through a vertical orifice under different heads, their velocities are not the same. In orifice flow, however, velocity is ordinarily taken as the velocity due to mean head H. The mean velocity thus obtained is represented by the symbol V_t, while the mean velocity in the channel of approach, called the *velocity of approach*, is represented by V_A. Considering points 1 and 2 as shown in Fig. 10.4 and applying the energy equation between them, we obtain

$$H + \frac{V_A^2}{2g} = \frac{V_t^2}{2g} \qquad (10.12)$$

or

$$V_t = \sqrt{2g\left[H + \frac{V_A^2}{2g}\right]} \qquad (10.13)$$

If the cross-sectional area of the vessel is large compared to the area of the orifice, the velocity of approach becomes negligible, and

$$V_t = \sqrt{2gH} \qquad (10.14)$$

In Eq.(10.13) the quantity in brackets and in Eq.(10.14) the quantity H represent the total head producing flow. Equation (10.14) indicates that the theoretical velocity of discharge from an orifice, i.e., the velocity which would exist if there were no loss of head, is the velocity acquired by a body falling freely in a vacuum through a height equal to the total head of the orifice. This principle; discovered by Torricelli in 1644, is known as the Torricelli's theorem.

Example 10.3

A water main gives a pressure reading of 490 kN/m². Find the theoretical rate at which water in it is escaped through a circular orifice 2.5 cm in diameter.

Solution We have, $p = 490$ kN/m², Diameter of orifice, $d = 2.5$ cm $= 0.025$ m

We know that, $p = \gamma H$

$$\therefore H = \frac{p}{\gamma} = \frac{490}{9.81} = 49.59 \text{ m of water}$$

Theoretical velocity of flow through the orifice

$$V_t = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 49.59} = 31.30 \; m/s$$

$\therefore$ Theoretical rate of flow at which water is escaped through the orifice

$$Q_t = A \times V_t = \frac{\pi}{4} \times 0.025^2 \times 31.30 = 0.015 \; m^3/s$$

Coefficient of Velocity

The actual velocity of the orifice jet at the vena contracta is less than the theoretical velocity, because of the frictional resistance that occurs as the liquid enters and passes through the orifice. The ratio of the actual mean velocity V of the jet at the vena contracta to the theoretical velocity V_t at the vena contracta that would occur without friction is called the coefficient of velocity and is designated by C_v. Thus,

$$C_v = \frac{V}{V_t} \quad \text{and} \quad V = C_v V_t \qquad (10.15)$$

The value of C_v varies from 0.95 to 0.99, the average value being 0.97. The value of C_v generally increases with the head under which the flow takes place.

Coefficient of Contraction

The ratio of the cross-sectional area of the jet at the vena contracta to the area of the orifice is called the coefficient of contraction. If a and A represent respectively the cross-sectional area of the jet at the vena contracta and the area of the orifice and C_c is the coefficient of contraction, then

$$C_c = \frac{a}{A} \quad \text{or} \quad a = C_c A \qquad (10.16)$$

The theoretical value of C_c is $\pi/(\pi + 2) = 0.611$, but in practice its value varies from 0.61 to 0.69, depending on the size and shape of the orifice and available head of the liquid under which the flow takes place. An average value of C_c is about 0.64.

Theoretical and Actual Discharges

If V_t be the theoretical velocity given by Eq.(10.14) and A be the cross-sectional area of the orifice, then the theoretical discharge through the orifice is given by

$$Q_t = AV_t = A\sqrt{2gH} \qquad (10.17)$$

On the other hand, if V be the actual mean velocity of the jet at the vena contracta, the actual discharge through the orifice is given by

$$Q = aV \qquad (10.18)$$

Since $a = C_c A$ and $V = C_v V_t$, the above equation becomes

$$Q = C_c A \times C_v V_t = C_c A \times C_v \sqrt{2gH} = C_c C_v A\sqrt{2gH} \qquad (10.19)$$

It is usual to replace the product C_cC_v with a single coefficient C_d, called the coefficient of discharge. The equation for the actual discharge of a fluid through an orifice thus becomes

$$Q = C_d A\sqrt{2gH} \tag{10.20}$$

Coefficient of Discharge

The ratio of the actual discharge through an orifice to the theoretical discharge is known as the coefficient of discharge C_d, i.e.

$$C_d = \frac{Actual\ discharge}{Theoretical\ discharge} = \frac{Q}{Q_t} \tag{10.21}$$

and comparing Eqs.(10.19) and (10.20), we obtain

$$C_d = C_c C_v \tag{10.22}$$

The value of C_d varies with the values of C_v and C_c. It varies from 0.59 to 0.65 depending upon the shape and size of the orifice, the head of liquid under which the flow takes place, the approach conditions and the viscosity of the fluid. An average value of C_d is about 0.62.

Head Lost in an Orifice

Orifice flow is no exception to the general rule that fluid motion is always accompanied by an expenditure of energy. For use in hydraulic engineering problems, the loss of energy due to flow through an orifice is conveniently expressed in two ways: (1) as a function of the velocity head, and (2) as a function of the total head.

1. Consider a fluid to be discharged from an orifice under a total head of H. The velocity of flow is $V = C_v\sqrt{2gH}$, from which the original head

$$H = \frac{1}{C_v^2}\frac{V^2}{2g} \tag{10.23}$$

The head remaining in the jet is the velocity head $V^2/2g$. So, the lost head H_0 = original head – remaining head, or

$$H_0 = H - \frac{V^2}{2g} = \frac{1}{C_v^2}\frac{V^2}{2g} - \frac{V^2}{2g} = \left(\frac{1}{C_v^2} - 1\right)\frac{V^2}{2g} \tag{10.24}$$

2. From Eq.(10.23), the velocity head in the jet

$$\frac{V^2}{2g} = C_v^2 H \tag{10.25}$$

Therefore, the lost head

$$H_0 = H - \frac{V^2}{2g} = H - C_v^2 H = (1 - C_v^2)H \tag{10.26}$$

The ratio of the loss of head in an orifice to the head of water available at the exit of the orifice is called the coefficient of resistance C_r, i.e.

$$C_r = \frac{Loss\ of\ head\ in\ the\ orifice}{Head\ of\ water\ available\ at\ the\ exit\ of\ the\ orifice} = \frac{H_0}{V^2/2g} \tag{10.27}$$

Using Eq.(10.24), we obtain

$$C_r = \frac{\left(\frac{1}{C_v^2} - 1\right)\frac{V^2}{2g}}{V^2/2g} = \frac{1}{C_v^2} - 1 \tag{10.28}$$

Experimental Determination of Orifice Coefficients

(1) Determination of the coefficient of discharge C_d: The coefficient of discharge C_d is of greatest value to the engineers. From Eqs. (10.17) and (10.21), we get

$$C_d = \frac{Actual\ discharge}{Theoretical\ discharge} = \frac{Q}{Q_t} = \frac{Q}{A\sqrt{2gH}} \qquad (10.29)$$

The simplest method of determining C_d is by measuring the actual discharge through the orifice, the area of the orifice and the total head, and then using Eq.(10.29).

(2) Determination of the coefficient of velocity C_v: The coefficient of velocity C_v can be obtained from the equation

$$C_v = \frac{V}{\sqrt{2gH}} \qquad (10.30)$$

if the velocity of the jet at the vena contracta V and the total head H are determined. The velocity of the jet at the vena contracta can be computed from the area of the jet and the measured discharge Q. It can also be determined with fair accuracy with a Pitot tube or by the coordinate method presented below.

The Coordinate Method of Determining C_v

Figure 10.5 represents a side view of a jet from an orifice. The jet at the vena contracta is traveling horizontally with a velocity V. The force of gravity causes the jet to move downward. Let x and y be the coordinates of any other point P in the jet. Neglecting air resistance, the horizontal component of the jet velocity is constant with the time t, from which

$$x = Vt$$

The jet has a downward acceleration which conforms to the law of falling bodies from which

$$y = \frac{1}{2}gt^2$$

Eliminating t between the above two equations

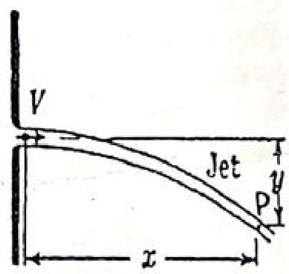


Fig. 10.5 Determination of C_v by coordinate method

$$x^2 = \frac{2V^2}{g}y \qquad (10.31)$$

which is the equation of a parabola with its vertex at the vena contracta. It is thus obvious that the path of a jet from a vertical orifice is a parabola.

Solving Eq.(10.31) for V, we get

$$V = \sqrt{\frac{gx^2}{2y}} \qquad (10.32)$$

Now, the theoretical velocity of jet is

$$V_t = \sqrt{2gH}$$

$$\therefore \text{ The coefficient of velocity, } C_v = \frac{V}{V_t} = \sqrt{\frac{x^2}{4yH}} \qquad (10.33)$$

(3) Determination of the coefficient of contraction C_c: The coefficient of contraction may be found by measuring the area of the jet at the vena contracta, and then by dividing the same by the area of the orifice, i.e.

$$\text{Coefficient of contraction, } C_c = \frac{Area\ of\ the\ jet\ at\ the\ vena\ contracta}{Area\ of\ the\ orifice} = \frac{a}{A} \qquad (10.34)$$

It is often difficult to locate the vena contracta exactly and measure the area of the jet at the vena contracta accurately. Therefore, it is recommended that the coefficient of contraction C_c be determined using the relation

$$C_c = \frac{C_d}{C_v} \tag{10.35}$$

Example 10.4

A 60 mm diameter orifice is discharging water under a head of 9 m. Calculate the actual discharge in liters per second and actual velocity of the jet at the vena contracta, if C_d = 0.62 and C_v = 0.96.

Solution Given, d = 60 mm = 0.06 m, H = 9 m, C_d = 0.62, C_v = 0.96

Area of the orifice, $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 0.06^2 = 0.00282\ m^2$

The theoretical discharge through the orifice is

$Q_t = A\sqrt{2gH} = 0.00282 \times \sqrt{2 \times 9.81 \times 9} = 0.0375\ m^3/s = 37.50\ liters/sec$

∴ The actual discharge through the orifice is

$Q_a = C_d Q_t = 0.62 \times 0.0375 = 0.02323\ m^3/s = 23.23\ liters/sec$

The theoretical velocity of the jet at vena contracta

$V_t = \sqrt{2gH} = \sqrt{2 \times 9.81 \times 9} = 13.29\ m/s$

∴ The actual velocity of the jet at vena contracta

$V_a = C_v V_t = 0.96 \times 13.29 = 12.76\ m/s$

Example 10.5

Water flows through a circular orifice 2.5 cm in diameter in the side of a tank. The constant head of water above the center of the orifice is 75 cm. The coordinates of the center line of the jet are 30 cm horizontally from the vena contracta and 3.2 cm vertically below the center of the orifice. The discharge from the orifice is 1.186 liters/sec. Find the orifice coefficients C_d, C_v and C_c.

Solution Given, d = 2.5 cm = 0.025 m, H = 75 cm = 0.75 m, x = 30 cm = 0.30 m, y = 3.2 cm =

0.032 m, Q = 1.186 liters/sec = 0.001186 m^3/s

Area of the orifice, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.025^2 = 0.000491\ m^2$

Coefficient of discharge, $C_d = \frac{Q}{Q_t} = \frac{Q}{A\sqrt{2gH}} = \frac{0.001186}{0.000491 \times \sqrt{2 \times 9.81 \times 0.75}} = 0.63$

Coefficient of velocity, $C_v = \sqrt{\frac{x^2}{4yH}} = \sqrt{\frac{0.30^2}{4 \times 0.032 \times 0.75}} = 0.97$

Coefficient of contraction, $C_c = \frac{C_d}{C_v} = \frac{0.63}{0.97} = 0.65$

10.5 MOUTHPIECES

It has been determined experimentally that if a short pipe is fitted to an orifice, it will increase the discharge through the orifice by increasing the value of the coefficient of discharge. Such a pipe, whose length is generally more than 2 times the diameter of the orifice and is fitted to the orifice externally or internally is known as a mouthpiece. The mouthpieces are used to increase the discharge by making them run full of water and increasing the coefficient of contraction.

The mouthpieces are classified based on the position, shape and nature of discharge as follows:

a) Position : External and internal mouthpieces

b) Shape : Cylindrical, convergent and convergent-divergent mouthpieces

c) Nature of discharge: Mouthpieces running full and running free

External Mouthpiece

The discharge through an orifice may be increased by fitting a short tube or pipe of sufficient length to the outside of the orifice as shown in Fig. 10.6. Such a tube or pipe which is attached externally to an orifice is known as the external mouthpiece. We know that the vena contracta occurs one-half diameter of the orifice away from the outlet of the orifice. If an external mouthpiece is fitted to the orifice and it has a length more than half the diameter, then the vena contracta will occur inside the mouthpiece. The minimum length of the mouthpiece is 2 to 3 times the diameter of the orifice. The jet will diverge after the vena contracta and thus the mouthpiece will be full with water, which makes the coefficient of contraction as unity. The coefficient of velocity is about 0.85 and hence the coefficient of discharge will be 0.85.

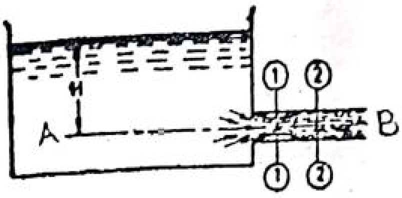


Fig. 10.6 External mouthpiece

Let V be the velocity of the liquid at outlet and V_c be the velocity of the liquid at the vena contracta. Assuming that the coefficient of contraction is 0.62, we have

$$a = C_c A = 0.62A \qquad (10.36)$$

As the flow is continuous, we have

$$aV_c = AV \qquad (10.37)$$

Combining the above two equations, we have

$$V_c = \frac{AV}{a} = \frac{V}{0.62} \qquad (10.38)$$

We see that the jet after contracting at section 1-1, suddenly enlarges at section 2-2. Due to this sudden enlargement, there will be a loss of head

$$h_e = \frac{(V_c - V)^2}{2g} = \frac{\left(\frac{V}{0.62} - V\right)^2}{2g} = 0.375\frac{V^2}{2g} \qquad (10.39)$$

Applying the energy equation between A and B, we have

$$H = \frac{V^2}{2g} + h_e = \frac{V^2}{2g} + 0.375\frac{V^2}{2g} = 1.375\frac{V^2}{2g} \qquad (10.40)$$

or

$$V = \sqrt{\frac{2gH}{1.375}} = 0.855\sqrt{2gH} \qquad (10.41)$$

We know that the theoretical velocity at the outlet

$$V_t = \sqrt{2gH}$$

$$\therefore \textit{Coefficient of velocity}, C_v = \frac{\textit{Actual velocity}}{\textit{Theoretical velocity}} = \frac{0.855\sqrt{2gH}}{\sqrt{2gH}} = 0.855 \qquad (10.42)$$

For finding the discharge through the external mouthpiece, we first determine the coefficient of discharge for the mouthpiece. Considering section 2-2, the coefficient of discharge

$$C_d = C_v \times C_c = 0.855 \times 1 = 0.855 \qquad (10.43)$$

This shows that the coefficient of discharge has considerably increased by fitting an external mouthpiece. The discharge through the mouthpiece is given by

$$Q = C_d A\sqrt{2gH} = 0.855 A\sqrt{2gH} \qquad (10.44)$$

It has been experimentally found that there is some loss of head at the entrance to the mouthpiece, depending upon the type of orifice. This loss of head sometimes reduces the coefficient of discharge up to 0.82. But for all practical purposes, the value of C_d is taken as 0.855.

The coefficient of discharge for an external mouthpiece also depends on its length. The coefficient of discharge will decrease with the increase in length due to greater frictional resistance in the mouthpiece. The following table shows the decrease in the coefficient of discharge with increase in the length of the mouthpiece.

Length of mouthpiece	3d	5d	10d	25d	50d
Coefficient of discharge	0.81	0.79	0.77	0.71	0.64

Example 10.6

Find out the discharge from a 100 mm diameter external mouthpiece, fitted to the sides of a large vessel, if the head over the mouthpiece is 4 m.

Solution Given, d = 100 mm = 0.10 m, H = 4 m

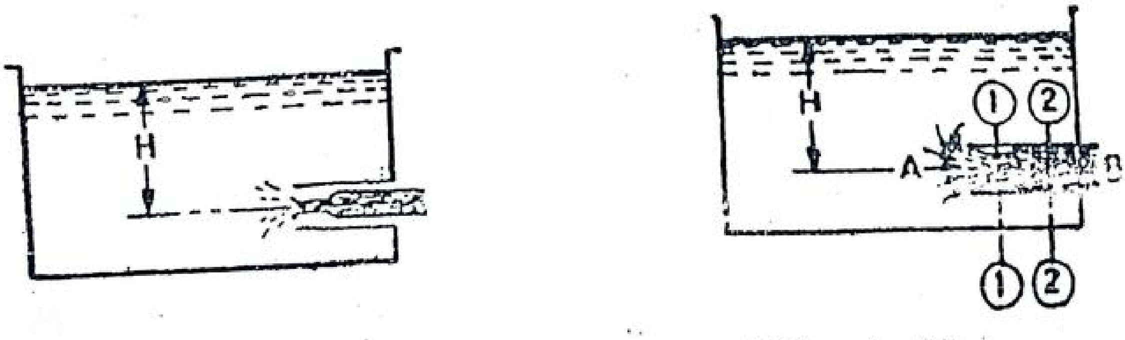
Area of the mouthpiece, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.10^2 = 0.007854\ m^2$

∴ Discharge from the mouthpiece

$$Q = 0.855 A\sqrt{2gH} = 0.855 \times 0.007854 \times \sqrt{2 \times 9.81 \times 5} = 0.06651\ m^3/s = 66.51\ liters/sec$$

Internal or Borda's Mouthpiece

A mouthpiece having its end projecting into the liquid, i.e. inside the vessel is known as an internal or re-entrant or inward-projecting mouthpiece (Fig. 10.7). It is also known as the Borda's mouthpiece after the French scientist Jean Charles Borda who showed that the coefficient of contraction for such a mouthpiece for an ideal fluid is 0.50.



(a) Running free (b) Running full

Fig. 10.7 Borda's mouthpiece

If the jet after contraction expands but does not touch the sides of the mouthpiece, it is said to be running free (Fig. 10.7a). But if the jet after contraction expands and fills up the whole mouthpiece, it is said to be running full (Fig. 10.7b). It has been found experimentally that, if the length of the mouthpiece is less than 3 times the diameter of the orifice, it will run free. But if the length of the mouthpiece is more than 3 times the diameter of the orifice, it will run full. The coefficient of discharge will be different in the two cases.

(1) Mouthpiece running free: With reference to Fig. 10.7a, let H be the head of liquid above the mouthpiece, a be the area of the contracted section, A be the area of the orifice or mouthpiece and V be the velocity of the jet at the contracted section.

Now, pressure of the liquid in the mouthpiece, $p = \gamma H$ and the force acting on the mouthpiece

$$= \text{pressure} \times \text{area} = \gamma H \times A = \gamma HA \qquad (10.45)$$

Mass of liquid flowing per sec

$$= \rho a V$$

∴ Momentum of the flowing liquid per second

$$= \text{Mass} \times \text{velocity} = \rho a V \times V = \rho a V^2 \qquad (10.46)$$

Since water is initially at rest, initial momentum = 0 and therefore the change of momentum

$$= \rho a V^2 \qquad (10.47)$$

According to Newton's second law of motion, the force is equal to the rate of change of momentum. Therefore, equating Eqs. (10.45) and (10.47), we get

$$\gamma HA = \rho a V^2 \quad \text{or} \quad HA = \frac{aV^2}{g} \quad (\because \gamma = \rho g)$$

or

$$\frac{V^2}{2g} \times A = \frac{aV^2}{g} \quad (\because \textit{for an ideal fluid, } H = \frac{V^2}{2g})$$

$$\therefore \quad A = 2a$$

$$\therefore \text{Coefficient of contraction, } C_c = \frac{a}{A} = \frac{a}{2a} = 0.5 \qquad (10.48)$$

For an ideal fluid, the coefficient of velocity, $C_v = 1$. So, the coefficient of discharge

$$C_d = C_v \times C_c = 1 \times 0.5 = 0.5 \qquad (10.49)$$

$$\therefore Q = C_d A\sqrt{2gH} = 0.5A\sqrt{2gh} \qquad (10.50)$$

Equation (10.50) gives the actual discharge through a Borda's mouthpiece when running free.

(2) Mouthpiece running full: With reference to Fig. 10.7b, let V_c be the velocity of the jet at the vena contracta and V be the velocity at the outlet. Since the flow is continuous

$$aV_c = AV$$

or

$$V_c = \frac{AV}{a} \qquad (10.51)$$

We have seen earlier that the coefficient of contraction for the Borda's mouthpiece is 0.5. Therefore, substituting a/A = 0.5, we obtain

$$V_c = 2V \qquad (10.52)$$

The jet after passing through section 1-1 suddenly enlarges at section 2-2. Therefore, there will be a loss of head due to sudden enlargement, given by

$$h_e = \frac{(V_c - V)^2}{2g} = \frac{(2V - V)^2}{2g} = \frac{V^2}{2g} \qquad (10.53)$$

Applying the energy equation between points A and B, we get

$$H = \frac{V^2}{2g} + h_e = \frac{V^2}{2g} + \frac{V^2}{2g} = \frac{V^2}{g}$$

$$\therefore V = \sqrt{gH} \qquad (10.54)$$

$\therefore$ Actual discharge, $Q = A\sqrt{gH}$. But the theoretical discharge, $Q_t = A\sqrt{2gH}$. Therefore, the coefficient of discharge

$$C_d = \frac{Actual\ discharge}{Theoretical\ discharge} = \frac{A\sqrt{gH}}{A\sqrt{2gH}} = \frac{1}{\sqrt{2}} = 0.707$$

$$\therefore Q = C_d A\sqrt{2gH} = 0.707 A\sqrt{2gH} \qquad (10.55)$$

Equation (10.55) gives the actual discharge through a Borda's mouthpiece when running full.

We see that the coefficient of discharge of an internal mouthpiece is less than that of an external mouthpiece. The reason is that in the case of external mouthpiece the liquid particles have to deviate through a maximum angle of 90^0. But in the case of internal mouthpiece, the liquid particles have to deviate by as much as 180^0. Due to more angle of deviation of the liquid particles, the contraction of the jet is more in the case of internal mouthpiece than that in the case of external mouthpiece. So, the coefficient of contraction C_c and the coefficient of discharge C_d for an internal mouthpiece are less than those for an external mouthpiece.

We also see that the discharge is more when a mouthpiece is running full than the discharge when a mouthpiece is running free. This is because when a mouthpiece is running full, negative pressure is created at the vena contracta which increases the discharge.

Example 10.7

A Borda's mouthpiece of 100 mm in diameter discharges water under a head of 4 m. Find the discharge in liters per second through the mouthpiece, when the mouthpiece is (a) running free, and (b) running full.

Solution We have, d = 100 mm = 0.1 m, H = 4 m

Area of the mouthpiece, $A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.10^2 = 0.007854\ m^2$

(a) When the mouthpiece is running free

$$Q = 0.5A\sqrt{2gH} = 0.5 \times 0.007854 \times \sqrt{2 \times 9.81 \times 4}$$

$= 0.0348\ m^3/s = 34.8$ liters/sec

(b) When the mouthpiece is running full

$$Q = 0.707A\sqrt{2gH} = 0.707 \times 0.007854 \times \sqrt{2 \times 9.81 \times 4}$$

$= 0.0492\ m^3/s = 49.2$ liters/sec

10.6 FLOW THROUGH NOZZLES

Introduction

A nozzle is a tapering mouthpiece, fitted to the outlet end of a pipe. It is generally used to have a high velocity of water, as it converts the total head of water into velocity or kinetic head. A high velocity of water is required in fire fighting, mining and power developments. Nozzles are used at the end of hose pipes and in some forms of turbines. As the pressure of the jet issuing from the nozzle is atmospheric, the whole of the energy is kinetic. The loss of energy in the nozzle itself is small compared with the frictional loss in the pipe to which the nozzle is fitted and may be neglected.

Velocity of Water through a Nozzle

Consider a nozzle BC fitted at the end of a pipe AB through which water is flowing (Fig.10.8). Let L, d and f are the length, diameter and friction factor for the pipe AB, V is the velocity of water in the pipe AB, v is the velocity of water through the nozzle, D is the diameter of the nozzle and H is the total head of water under which the flow takes place.

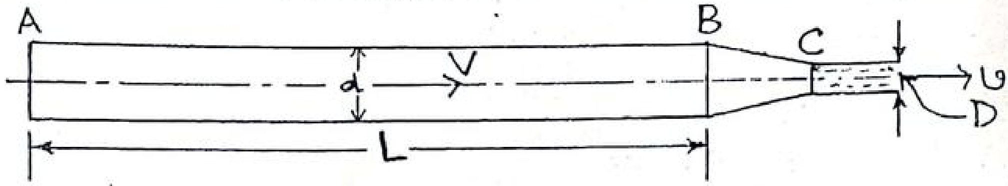


Fig. 10.8 A nozzle

Then, the area of the pipe AB is

$$A = \frac{\pi}{4} d^2 \tag{10.56}$$

and the area of the nozzle at C is

$$a = \frac{\pi}{4} D^2 \tag{10.57}$$

Since the flow is continuous, the discharge

$$Q = AV = av \qquad \text{or,} \qquad V = \frac{av}{A} \tag{10.58}$$

We know that the loss of head due to friction in the pipe AB

$$h_f = f \frac{L}{d} \frac{V^2}{2g} \tag{10.59}$$

and the loss of head due to velocity at outlet

$$= \frac{v^2}{2g} \tag{10.60}$$

Assuming that the minor losses are negligible and the total available head of water is lost while flowing through the pipe and the nozzle, we can write

H = Loss of head due to friction + Loss of head due to velocity at outlet

$$= h_f + \frac{v^2}{2g} \tag{10.61}$$

Using Eqs.(10.58) and (10.59), Eq. (10.61) becomes

$$H = f \frac{L}{d} \frac{V^2}{2g} + \frac{v^2}{2g} = f \frac{L}{d.2g} \left(\frac{a^2 v^2}{A^2} \right) + \frac{v^2}{2g} = \frac{v^2}{2g} \left(f \frac{L}{d} \times \frac{a^2}{A^2} + 1 \right) \tag{10.62}$$

$$\therefore v = \sqrt{\frac{2gH}{1 + f\frac{L}{d} \times \frac{a^2}{A^2}}} \tag{10.63}$$

Equation (10.63) gives the velocity of water through a nozzle.

Example 10.8

A pipe 3.2 km long and 90 cm in diameter is fitted with a nozzle of 20 cm diameter at the discharge end. Find the velocity of water through the nozzle if the head of water is 50 m. Take f = 0.024 for the pipe.

Solution Here, L = 3.2 km = 3200 m, d = 90 cm = 0.90 m, D = 20 cm = 0.20 m, H = 50 m, f = 0.024

Area of the pipe, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 0.90^2 = 0.6362\ m^2$

Area of the nozzle, $a = \frac{\pi}{4} D^2 = \frac{\pi}{4} \times 0.20^2 = 0.0314\ m^2$

Using Eq.(10.63), we obtain

$$v = \sqrt{\frac{2gH}{1 + f\frac{L}{d} \times \frac{a^2}{A^2}}} = \sqrt{\frac{2 \times 9.81 \times 50}{1 + 0.024 \times \frac{3200}{0.90} \times \frac{0.0314^2}{0.6362^2}}} = 68.3\ m/s$$

Transmission of Power through a Nozzle

We know that the kinetic energy of the jet through the nozzle

$$= \frac{\gamma Q v^2}{2g} \text{ m-kg/s}$$

$\therefore$ Power available at the outlet of the jet

$$P = \frac{\gamma Q v^2}{2g \times 75}\ h.p. \qquad (\because 1 \text{ h.p.} = 75 \text{ m-kg/s}) \tag{10.64}$$

Using Eq.(10.61), we obtain

$$\frac{v^2}{2g} = H - h_f \tag{10.65}$$

$\therefore$ Power transmitted through the nozzle is given by

$$P = \frac{\gamma Q(H - h_f)}{75}\ h.p. \tag{10.66}$$

$$= \frac{\gamma Q}{75}\left(H - f\frac{L}{d}\frac{V^2}{2g}\right) \tag{10.67}$$

and using Eq.(10.58)

$$P = \frac{\gamma a v}{75}\left[H - \frac{fL}{2gd}\left(\frac{a^2 v^2}{A^2}\right)\right]\ h.p. \tag{10.68}$$

Example 10.9

Water is supplied to a hydro-electric plant at the rate of 500 liters/sec under a head of 250 m through a pipeline 3.2 km long and 500 mm in diameter. The pipeline ends in a nozzle of diameter 200 mm. Find the power that can be transmitted, if f for the pipe is 0.040.

Solution We have, Q = 500 liters/sec = 0.50 m^3/s, H = 250 m, L = 3.2 km = 3200 m, d = 500 mm = 0.50 m, D = 200 mm = 0.20 m, f = 0.040

Area of the pipe, $A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.50^2 = 0.1963\ m^2$

Velocity of water through the pipe, $V = \frac{Q}{A} = \frac{0.50}{0.1963} = 2.547\ m/s$

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = 0.040 \times \frac{3200}{0.50} \times \frac{2.547^2}{2 \times 9.81} = 84.6526\ m$$

Using Eq.(10.66), the power that can be transmitted is given by

$$P = \frac{\gamma Q(H - h_f)}{75} = \frac{1000 \times 0.50 \times (250 - 84.6526)}{75} = 1102.32\ h.p.$$

Efficiency of Power Transmission through a Nozzle

We know that power available at the outlet of the nozzle

$$P = \frac{\gamma Q v^2}{2g \times 75}$$

and power available at the inlet of the pipe

$$= \frac{\gamma QH}{75}$$

∴ Efficiency of power transmission

$$\eta = \frac{\text{Power available at the outlet of the nozzle}}{\text{Power available at the inlet of the pipe}} = \frac{\frac{\gamma Q v^2}{2g \times 75}}{\frac{\gamma QH}{75}} = \frac{v^2}{2gH} \qquad (10.69)$$

Condition for Maximum Transmission of Power

The power transmitted will be maximum when dP/dv = 0. From Eq.(10.68) it is seen that dP/dv will be zero when

$$\frac{d}{dv}\left[\frac{\gamma av}{75}\left(H - \frac{fL}{2gd} \times \frac{a^2v^2}{A^2}\right)\right] = 0$$

or,

$$\frac{d}{dv}\left[\frac{\gamma a}{75}\left(Hv - \frac{fL}{2gd} \times \frac{a^2v^3}{A^2}\right)\right] = 0$$

or,

$$H - 3\left(\frac{fL}{2gd} \times \frac{a^2v^2}{A^2}\right) = 0$$

or,

$$H - 3\left(f\frac{L}{d}\frac{V^2}{2g}\right) = 0 \qquad \left(\because V = \frac{av}{A}\right)$$

or,

$$H - 3h_f = 0 \qquad \left(h_f = f\frac{L}{d}\frac{V^2}{2g}\right)$$

$$\therefore h_f = \frac{H}{3} \qquad (10.70)$$

This means that the power transmitted through the nozzle is maximum when the loss of head due to friction in the pipe is equal to 1/3 of the total supply head.

Example 10.10

A pipe 75 mm in diameter and 250 m long has a nozzle of 25 mm in diameter fitted at the discharge end. If the total head of water is 40 m, find the maximum power transmitted. Take f = 0.040.

Solution Here, d = 75 mm = 0.075 m, L = 250 m, D = 25 mm = 0.025 m, H = 40 m, f = 0.040

Area of the pipe, $A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times 0.075^2 = 0.0044\ m^2$

We know that for maximum transmission of power, the loss due to friction is 1/3 of the total head.

$$\therefore h_f = \frac{H}{3} = \frac{40}{3} m$$

Now,

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = f\frac{L}{d}\frac{Q^2}{2gA^2}$$

or,

$$\frac{40}{3} = 0.040 \times \frac{250}{0.075} \times \frac{Q^2}{2 \times 9.81 \times 0.0044^2}$$

from which we obtain Q = 0.0061 m^3/s. Now, using Eq.(10.66)

$$P = \frac{\gamma Q(H - h_f)}{75} = \frac{1000 \times 0.0061 \times \left(40 - \frac{40}{3}\right)}{75} = 2.17\ h.p.$$

Water Hammer

If water, flowing in a long pipe, is suddenly brought to rest by closing the valve, its momentum is destroyed which causes a very high pressure on the valve. This high pressure is followed by a series of pressure vibrations. These pressure vibrations set up noises in the pipe, known as knocking. Such a knowing is often heard in the water pipes of ordinary dwelling houses if the tap is turned off quickly. The sudden rise of pressure has the effect of hammering action on the walls of the pipe, and thus is known as the hammer blow or water hammer. Sometimes, the hammer blow is so high that it may even burst the pipe. It is thus obvious that the valves of the pipelines should always be closed gradually.

10.7 WEIRS

Description

A weir (Fig. 10.9) is an overflow structure built across an open channel to measure the flow. Weirs may be rectangular, triangular, trapezoidal, circular, parabolic or any other regular form. The edge or top surface with which the flowing liquid comes in contact is termed the crest of the weir. Classified with reference to the form of the crest, weirs may be sharp-crested or broad-crested. The sharp-crested weir has a sharp upstream edge so formed that the liquid in passing touches only a line.
The broad-crested weir has either a sharp upstream edge or a crest so broad that the liquid in passing comes in contact with a surface.

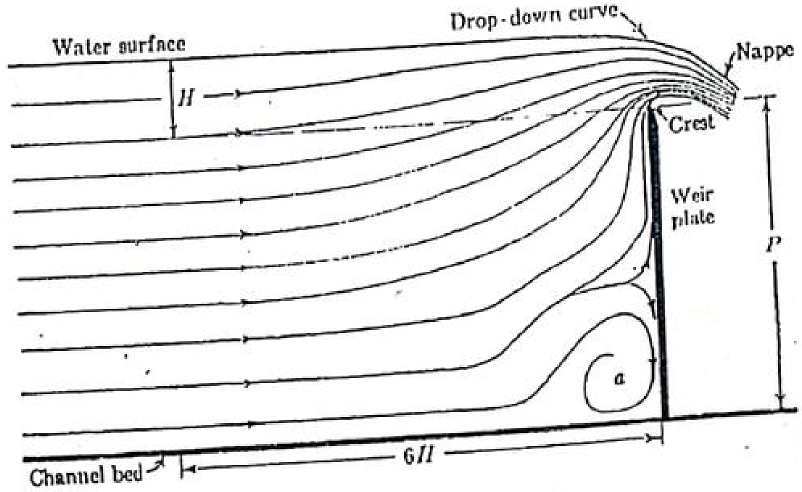


Fig. 10.9 Flow over a sharp-crested weir

The flow over a weir may be either free or submerged. If the water surface downstream from the weir is lower than the crest, the flow is free. If the downstream water surface is higher than the crest, the flow is submerged.

The overflowing stream is termed the nappe. The nappe of a sharp-crested weir is contracted at its underside by the action of the vertical component of velocity just upstream from the weir. This is called vertical or crest contraction. If the sides of the weir also have sharp upstream edges so that the nappe is contracted in width, the weir is said to have end contractions. A weir having its width (transverse to the flow) equal to the width of the channel, so that only vertical contraction of the nappe takes places, is called a *suppressed or full-width weir*. When the width of the weir is less than the width of the channel so that the nappe contracts both in the vertical and lateral directions, the weir is termed a *contracted weir*.

There is a downward curvature of the surface of the liquid in the vicinity of the weir which is called the drop-down curve. The vertical distance H between the liquid surface and the crest of the weir, measured far enough upstream to be beyond the drop-down curve, is called the head, and the mean velocity in this channel is called the *velocity of approach*. The height P of the weir is the vertical distance of the crest above the bottom of the channel.

Typical path lines of flow over a sharp-crested weir are shown in Fig. 10.9. The paths are approximately parallel until they reach a point about 6 times the head upstream from the weir. From this point they gradually curve upward to pass over the crest. There is a dead water region at the corner just upstream of the weir.

The crest of a sharp-crested weir is not necessarily knife-edged. In practice, the crest length of 1 mm to 2 mm in the direction of flow is provided and the downstream end is beveled at an angle of 45° to 60°. In this case, the flow springs clear of the weir body downstream of the weir and an air pocket is formed beneath the nappe from which air is continuously removed by the overflowing jet. As the flow continues, the pressure in the air pocket falls below the atmospheric pressure and the nappe is depressed. For flow measurement, the atmospheric pressure is maintained in the air pocket through the provision of air vents (Fig. 10.10).

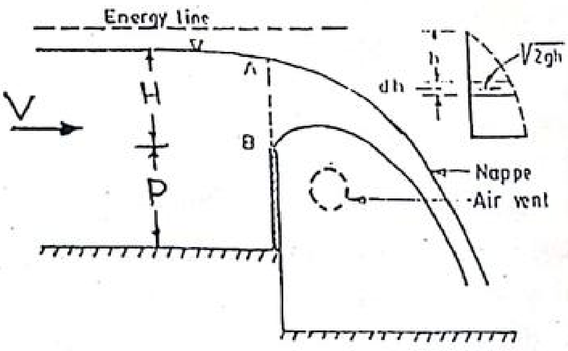


Fig. 10.10 Free flow over a rectangular sharp-crested suppressed weir

Rectangular sharp-crested suppressed weir

Free flow: Consider a rectangular sharp-crested weir spanning the full width b of a rectangular channel as shown in Fig. 10.10. It is assumed that the flow does not contract as it passes over the weir and the pressure is atmospheric across the whole section AB. The velocity at any depth h below the energy line is equal to $\sqrt{2gh}$ and the discharge through an elementary strip of thickness dh is given by

$$dQ = b\sqrt{2gh}\,dh \tag{10.71}$$

The total discharge Q is then

$$Q = b\sqrt{2g}\int_{V^2/2g}^{H+V^2/2g}\sqrt{h}\,dh = \frac{2}{3}\sqrt{2g}\,b\left[\left(H+\frac{V^2}{2g}\right)^{1.5}-\left(\frac{V^2}{2g}\right)^{1.5}\right] \tag{10.72}$$

$$= \frac{2}{3}\sqrt{2g}\,bH^{1.5}\left[\left(1+\frac{V^2}{2gH}\right)^{1.5}-\left(\frac{V^2}{2gH}\right)^{1.5}\right] \tag{10.73}$$

where

$$V = \frac{Q}{b(H+P)} \tag{10.74}$$

is the approach velocity.

The effect of flow contraction is taken into account by a coefficient of contraction C_c. Then

$$Q = \frac{2}{3}C_c\sqrt{2g}\,bH^{1.5}\left[\left(1+\frac{V^2}{2gH}\right)^{1.5}-\left(\frac{V^2}{2gH}\right)^{1.5}\right] \tag{10.75}$$

Introducing a discharge coefficient C_d, Eq.(10.75) can be written in a more compact form as

$$Q = \frac{2}{3}C_d\sqrt{2g}\,bH^{1.5} \tag{10.76}$$

where

$$C_d = C_c\left[\left(1+\frac{V^2}{2gH}\right)^{1.5}-\left(\frac{V^2}{2gH}\right)^{1.5}\right] \tag{10.77}$$

If the Reynolds number of the flow is sufficiently high and the upstream depth H is at least 0.11 m so that the surface tension and viscosity effects are negligible, then C_d becomes independent of the Reynolds and Weber numbers and depends only on the ratio H/P. The variation of C_d for rectangular sharp-crested weirs is given with satisfactory accuracy using the well-known *Rehbock formula*

$$C_d = 0.611 + 0.08\,H/P \tag{10.78}$$

which is valid for $H/P \le 5$.

When P becomes very large, C_d becomes equal to 0.611. Since in this case $V^2/2gH$ becomes negligibly small, Eq.(10.77) shows that C_c also becomes equal to 0.611.

Submerged flow: The discharge over a broad-crested weir is affected by the tailwater level downstream of the weir if it is above the weir crest. Such a flow is called a submerged flow. Under submerged conditions, the discharge over the weir depends on the submergence ratio H_2/H_1 (Fig. 10.11), and is given by the *Villemonte equation*

$$Q_s = Q\left[1-\left(\frac{H_2}{H_1}\right)^n\right]^{0.385} \qquad (10.79)$$

where Q is the free-flow discharge under head H_1 (Eq.(10.76) and n is the exponent of head in the head-discharge relationship $Q = kH^n$. For rectangular weirs, n = 1.5 and for triangular weirs, n = 2.5.

To ensure free flow over a broad-crested weir, the water level downstream of the weir must be kept a few centimeters below the weir crest.

Rectangular sharp-crested contracted weir

In a contracted weir (Fig.10.12), the effective width of the weir (transverse to the direction of flow) is reduced and is given by the well-known *Francis formula*

$$B_e = B - 0.1nH \qquad (10.80)$$

where B is the width of the weir and n is the number of end contractions. The discharge equation for a contracted weir may be developed in the same way as for a suppressed weir. However, Kindsvater and Carter (1957), based on their extensive experimental investigation, modified the theoretical equation so that it would apply to all rectangular sharp-crested weirs regardless of whether they are suppressed or contracted. The equation for discharge over a sharp-crested weir is

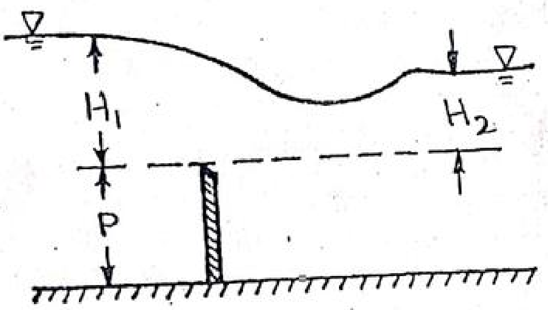


Fig. 10.11 Submerged sharp-crested weir

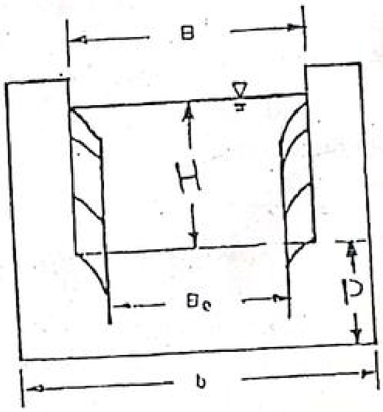


Fig. 10.12 Weir with end contractions

$$Q = \frac{2}{3} C_{de} \sqrt{2g}\, B_e H_e^{1.5} \qquad (10.81)$$

where C_{de} is the effective coefficient of discharge and B_e and H_e are the effective width and head of the weir, and

$$C_{de} = K_1 + K_2 \times \frac{H}{P} \qquad (10.82a)$$

$$B_e = B + K_b \qquad (10.82b)$$

$$H_e = H + K_h \qquad (10.82c)$$

where the parameters K_b and K_h represent the combined effects of viscosity and surface tension on the flow. Usually K_h is taken to be equal to 0.001 m. The parameters K_1, K_2 and K_b, which depend on B/b, are given in Table 10.1.

Table 10.1 Values of K_1, K_2 and K_b for broad-crested weirs (Kindsvater and Carter, 1957)

B/b	K_1	K_2	K_b
1.0	0.602	0.0750	-0.0009
0.9	0.599	0.0640	0.0037
0.8	0.597	0.0450	0.0043
0.7	0.595	0.0300	0.0041
0.6	0.593	0.0180	0.0037
0.5	0.592	0.0110	0.0030
0.4	0.591	0.0058	0.0027
0.3	0.590	0.0020	0.0025
0.2	0.589	-0.0018	0.0024
0.1	0.588	-0.0021	0.0024

Example 10.11

A rectangular sharp-crested weir spanning the full width of a rectangular channel 2 m wide is 1 m high. Compute the discharge over the weir under an upstream head of 0.75 m.

Solution Rectangular channel, b = 2 m, P = 1 m, H = 0.75 m

$C_d = 0.611 + 0.08H/P = 0.611 + 0.08 \times 0.75/1 = 0.671$

$$\therefore Q = \frac{2}{3} C_d \sqrt{2g}\, bH^{1.5} = \frac{2}{3} \times 0.671 \times \sqrt{2 \times 9.81} \times 2 \times 0.75^{1.5} = 2.574\, m^3/s$$

Example 10.12

Compute the discharge over a sharp-crested contracted weir 1 m wide and 1 m high set in a rectangular channel 2 m wide if the head over the weir is 0.75 m.

Solution We have, b = 2 m, B = 1 m, P = 1 m, H = 0.75 m, K_h = 0.001 m

$$\frac{B}{b} = \frac{1}{2} = 0.50$$

$\therefore$ From Table 10.1, $K_1 = 0.592$, $K_2 = 0.0110$ and $K_b = 0.0030$

$$\therefore C_{de} = K_1 + K_2 \times \frac{H}{P} = 0.592 + 0.0110 \times \frac{0.75}{1} = 0.60025$$

$B_e = B + K_b = 1 + 0.0030 = 1.0030$ m

$H_e = H + K_h = 0.75 + 0.001 = 0.751$ m

$$\therefore Q = \frac{2}{3} C_{de} \sqrt{2g}\, B_e H_e^{1.5} = \frac{2}{3} \times 0.60025 \times \sqrt{2 \times 9.81} \times 1.0030 \times 0.751^{1.5} = 1.161\, m^3/s$$

Triangular Weirs

The triangular weir is also known as the V-notch weir. Figure 10.13 represents a triangular weir over which a liquid is flowing. The measured head, i.e. the height of the liquid surface above the apex of the weir is H. The sides make equal angles with the vertical.

From the geometry of the figure, we find that the width of the weir at the water surface

$= 2H \tan \theta/2$

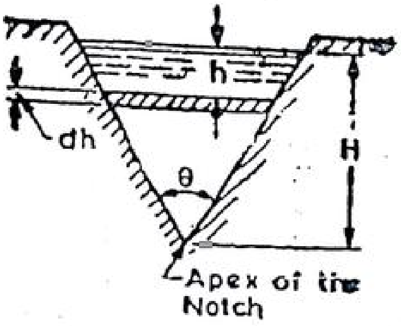


Fig. 10.13 Triangular or V-notch weir

and the area of the strip

$= 2(H - h) \tan \theta/2 \times dh$

Neglecting the velocity of approach and the friction loss, the theoretical velocity of water through the strip is $\sqrt{2gh}$ and the theoretical discharge is

$$dQ_t = 2(H-h)\tan\frac{\theta}{2} \times dh \times \sqrt{2gh} \qquad (10.83)$$

The total discharge over the entire weir may be obtained by integrating the above equation between the limits 0 and H.

$$\therefore Q_t = \int_0^H 2(H-h)\tan\frac{\theta}{2}\sqrt{2gh}\ dh = \frac{8}{15}\sqrt{2g}\tan\frac{\theta}{2} H^{5/2} \qquad (10.84)$$

Introducing a discharge coefficient C_d, the actual discharge

$$Q = \frac{8}{15} C_d \sqrt{2g}\tan\frac{\theta}{2} H^{2.5} \qquad (10.85)$$

The most common angle of a triangular weir is 90^0 for which $C_d = 0.60$. Hence, the formula for discharge becomes

$$Q = 1.417 H^{2.5} \qquad (10.86)$$

Example 10.13

A right-angled triangular weir is used to measure the discharge in an open channel. If the depth of water is 200 mm, calculate the discharge over the weir in liters per sec. Assume $C_d = 0.62$.

Solution We have, $\theta = 90^0$, H = 200 mm = 0.20 m, $C_d = 0.62$

$$\therefore \quad Q = \frac{8}{15} C_d \sqrt{2g}\tan\frac{\theta}{2} H^{2.5} = \frac{8}{15} \times 0.62 \times \sqrt{2 \times 9.81} \times \tan 45^0 \times 0.20^{2.5}\ m^3/s$$

$= 0.0262\ m^3/s = 26.2$ liters/sec

PROBLEMS AND EXERCISES

10.1 What is the use of a Pitot tube? What is the basic principle on which a Pitot tube works? How can a Pitot tube be used to obtain the velocity head in a closed pipe?

10.2 What is the use of a Venturimeter? State the principle on which a Venturimeter works.

10.3 What is the use of an orifice? State the principle on which an orifice works.

10.4 What do you mean by the coefficient of contraction, the coefficient of velocity and the coefficient of discharge of an orifice? What are their ranges of values and the average values? Describe how can you determine them experimentally.

10.5 How does a mouthpiece increase the discharge of an orifice? Show that the coefficient of discharge for an external mouthpiece is 0.855.

10.6 What is a Borda's mouthpiece? Show that the coefficient of discharge for a Borda's mouthpiece is (i) 0.50 when running free, and (ii) 0.707 when running full.

10.7 Explain why (i) the coefficient of discharge for an external mouthpiece is more than that of an internal mouthpiece, and (ii) the discharge is more when a mouthpiece is running full than the discharge when the mouthpiece is running free.

10.8 What is a nozzle? State its uses. Derive the condition for maximum transmission of power through a nozzle.

10.9 What is the use of a weir? What is the difference between (i) sharp-crested and broad-crested weirs, (ii) free and submerged weirs, (iii) suppressed and contracted weirs?

10.10 What do we get from (i) the Rehbock formula, (ii) the Villemonte formula, (iii) the Francis formula, and (iv) the Kindsvater and Carter formula?

10.11 Derive the formula for theoretical discharge over a triangular or V-notch weir.

10.12 A Pitot tube is placed in water as shown in Fig. 10.1 and the velocity of stream upstream of the tube is 2 m/s. How much the water will rise in the tube above the free water surface?

10.13 A submarine moves horizontally in sea and has its axis 15 m below the surface of water. A Pitot tube properly placed just in front of submarine and along its axis is connected to the two limbs of a U-tube containing Hg. The difference in Hg level is found to be 17 cm. Find the speed of the submarine, if the density of Hg is 13.6 and that of sea water is 1.026.

10.14 Calculate the flow of water in liters/sec through a 40 cm × 15 cm horizontal Venturimeter, when the differential gauge connected to the inlet end of the meter and its throat shows 25 cm of Hg. Assume the discharge coefficient as 0.98.

10.15 A Venturimeter having a throat diameter of 15 cm is connected to a 30 cm diameter pipe running full of water. It is laid in an inclined position. An inclined U-tube manometer with measuring liquid of specific gravity 0.6 is used to measure pressure difference. Calculate the rate of flow in the pipe if the reading given by the manometer is 30 cm and the losses between inlet and throat is 0.2 time the velocity head in the pipe.

10.16 A circular orifice 4 cm in diameter is discharging water under a head of 3 m. Compute (i) the theoretical discharge through the orifice, (ii) the actual discharge through the orifice, (iii) the theoretical velocity of the jet at the vena contracta, (iv) the actual velocity of the jet at the vena contracta, and (v) the diameter of the jet at the vena contracta. Take $C_c = 0.64$ and $C_v = 0.97$ for the orifice.

10.17 A circular orifice 4 cm in diameter is discharging water under a head of 6 m. The actual discharge through the orifice is 516 liters per minute. The coordinates of the center line of the jet are 120 cm horizontally and 6.40 cm vertically below the center line of the orifice. Compute the orifice coefficients C_c, C_v and C_d.

10.18 The actual discharge through an external mouthpiece 9 cm in diameter, fitted to the sides of a large tank, is 60 liters/sec. Determine the head over the mouthpiece.

10.19 An internal mouthpiece of 100 mm diameter discharges water under a head of 4 m. Taking C_c equal to 0.64, calculate the coefficient of discharge, when the mouthpiece is running full and the only loss is due to sudden enlargement.

10.20(a) An external mouthpiece of 5 cm in diameter discharges water under a head of 5 m. Find the discharge in liters/sec.

(b) A Borda's mouthpiece of 5 cm in diameter discharges water under a head of 5 cm. Find the discharge in liters/sec when the mouthpiece is (i) running free, and (ii) running full.

10.21 A pipe 100 mm in diameter and 1 km long is fitted with a nozzle 25 mm in diameter at its discharge end. The total head of water is 50 m and the friction factor for the pipe is 0.030. Determine (i) the velocity of flow through the nozzle, (ii) the velocity of flow through the pipe, (iii) the power that can be transmitted, (iv) the efficiency, (v) the maximum power that can be transmitted, and (vi) the efficiency for the maximum power transmission.

10.22 A rectangular sharp-crested weir spanning the full width of a rectangular channel 3 m wide is 1 m high. Compute the discharge over the weir under an upstream head of 0.50 m.

10.23 Compute the discharge over a rectangular sharp-crested contracted weir 3 m wide and 1 m high set in a rectangular channel 4 m wide if the head over the weir is 0.50 m.

10.24 Compute the discharge in liters per minute over a 90^0 V-notch weir under a head of 250 mm if $C_d = 0.60$.

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ANSWERS TO THE PROBLEMS

Chapter 1

1.7(i) 998.23 kg/m^3 (ii) 9792.64 N/m^3 (iii) 0.001 $N\ s/m^2$ (iv) $1.002 \times 10^{-6}\ m^2/s$
(v) $2.19 \times 10^9\ N/m^2$ (vi) 0.0728 N/m
1.8 $5.88 \times 10^{-4}\ m^2/s$ 1.9 200 MN/m^2 1.10 -2.52 mm
1.11 (a) $0.74 \times 10^{-4}\ m^2/s$ (b) 0.65 N/m^2 (c) 0.65 N/m^2 (d) On the upper plate in the negative x direction and on the lower plate in the positive x direction
1.12 -0.004 N/m^2 (in the negative x direction), 1.256 N
1.13 du/dy = 0, 1.5,3, 4.5, 6 /s and τ = 0, 0.0015, 0.003, 0.0045, 0.006 N/m^2

Chapter 2

2.7 10.19 m of water 2.8 9.81 kN/m^2 2.9 (i) 10.34 m of water (ii) 101.40 kN/m^2
2.10 (i) 147.15 kN/m^2 (ii) 11557 kN (iii) p_{max} = 147.15 kN/m^2, p_{min} = 0, p_{avg} = 73.57 kN/m^2
(iv) 34,670 kN
2.11 42.90 cm 2.12 60.04 kN 2.14 4.205 m 2.15 2330 kg, 49° 57′
2.16 P_H = 25000 kg, P_V = 7140 kg 2.17 57220 kg, 51° 52′ 2.18 90863 kg (891.2 kN), 37° 36′

Chapter 3

3.5 222.22 kg 3.6(i) 0.2 m^3 (ii) 0.33 m^3
3.7 GM = -0.95 m, Cannot float vertically (unstable equilibrium) 3.8 1.236 m

Chapter 4

4.5(a) 1720 liters (b) 27.59 kN, 13.26 kN 4.6 2.815 kg/m^2
4.7(a)(i) 187.62 kN, (ii) 62.54 kN (b)(i) 250.155 kN, (ii) 0 4.8 12.46 kN/m^2, 16.97 kN/m^2
4.9 21.41 m

Chapter 5

5.4 10 m/s 5.5 Irrotational 5.6 16 m/s^2, 1 m/s^2

Chapter 6

6.7 0.318 m^3/s, 4.5 m/s, 10.86 m/s 6.9 $\omega = -4xz - 2yz - 2z^3/3 + f(x,y,t)$
6.10 10,000 kg/m^2 (1 kg/cm^2) 6.11 2.70, 1.50 6.12 0.34 m of water, 5.92 kN
6.13 15.93 m, 0, 44194 N 6.14 $2\rho A V^2$

Chapter 7

7.5 (a) Dimensionally homogeneous (b) Not dimensionally homogeneous
7.7 $Q = C(\Delta p/l)(D^4/\mu)$ 7.8 13 m/s 7.9 20 m/s, 5.32 m 7.10 0.00082 kg/cm^2

Chapter 8

8.9 Laminar 8.10 18.276 m 8.12 66.4 liters/sec
8.13 0.027 8.14 0.026 8.15 4 cm

Chapter 9

9.7 0.206 m^3/s 9.8 12.659 m 9.9(a) 12.50 m (b) 6755.47 m (c) 37.72 cm
9.10 226 m^3/s 9.11 0.28 m^3/s 9.12 20.51 m 9.13(a) 1704.30 liters/min (b) 445.50 liters/min
9.14 After the second adjustment 59.08 m^3/s and 1.97 m^3/s clockwise and 41.92 m^3/s anticlockwise in the first loop and 31.22 m^3/s clockwise and 1.97 m^3/s and 43.89 m^3/s anticlockwise in the second loop 9.15 Q_1 = 0.54 m^3/s, Q_2 = 0.42 m^3/s and Q_3 = 0.12 m^3/s

Chapter 10

10.12 29 cm 10.13 23 km/hour 10.14 0.137 liters/sec 10.15 27.8 liters/sec
10.16(i) 0.009641 m^3/s (ii) 0.0060 m^3/s (iii) 7.67 m/s (iv) 7.44 m/s (v) 3.2 cm
10.17 C_c = 0.65, C_v = 0.97, C_d = 0.63 10.18 6.205 m 10.19 0.87
10.20(i) 16.63 liters/sec (b)(i) 9.73 liters/sec (ii) 13.75 liters/sec
10.21(i) 7.048 m/s (ii) 0.4405 m/s (iii) 2.170 h.p. (iv) 5.06% (v) 3.644 h.p. (vi) 28.44%
10.22 2.039 m^3/s 10.23 1.934 m^3/s 10.24 2657.67 liters/minute

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